S. P. Rigaud
april 15. 1815

AN

EASY INTRODUCTION

TO THE

MATHEMATICS;

IN WHICH

THE THEORY AND PRACTICE

ARE LAID DOWN AND FAMILIARLY EXPLAINED.

To each subject are prefixed,

A BRIEF POPULAR HISTORY OF ITS RISE AND PROGRESS, CONCISE MEMOIRS OF NOTED MATHEMATICAL AUTHORS ANCIENT AND MODERN, AND SOME ACCOUNT OF THEIR WORKS.

The whole forming

A COMPLETE AND EASY SYSTEM

OF

ELEMENTARY INSTRUCTION

IN THE

LEADING BRANCHES OF THE MATHEMATICS;

DESIGNED TO FURNISH STUDENTS WITH THE MEANS OF ACQUIRING CONSIDERABLE PROFICIENCY, WITHOUT THE NECESSITY OF VERBAL ASSISTANCE.

Adapted to the use of

SCHOOLS, JUNIOR STUDENTS AT THE UNIVERSITIES, AND PRIVATE LEARNERS,

ESPECIALLY THOSE WHO STUDY WITHOUT A TUTOR.

IN TWO VOLUMES.

BY CHARLES BUTLER.

Segniùs irritant animos demissa per aurem,
Quàm quæ sunt oculis subjecta fidelibus, et quæ
Ipse sibi tradit spectator. HOR.

Longum iter est per præcepta, breve et efficax per exempla. SENECA.

VOL. I.

OXFORD:
PRINTED BY BARTLETT AND NEWMAN;
AND SOLD BY LONGMAN, HURST, REES, ORME, AND BROWN, PATERNOSTER ROW, LONDON; PARKER, OXFORD; AND DEIGHTON, CAMBRIDGE.

1814.

TO THE

REV. JAMES WILDING, M. A.

MASTER OF CHEAM SCHOOL,

AND LATE FELLOW OF MAGDALEN COLLEGE, CAMBRIDGE,

AS A TESTIMONY OF ESTEEM FOR HIS VIRTUES,

OF RESPECT FOR HIS SUPERIOR TALENTS,

AND INDEFATIGABLE ZEAL

AS A PRECEPTOR;

AND OF GRATITUDE FOR REPEATED FAVOURS.

ALSO TO THE

NOBLEMEN AND GENTLEMEN

EDUCATED AT CHEAM SCHOOL,

AMONG WHOM THE AUTHOR HAS LABOURED NEARLY THIRTY YEARS;

TO EVINCE HIS SENSE OF THEIR GENERAL GOOD CONDUCT,

AND DILIGENCE IN MATHEMATICAL STUDIES;

AND LIKEWISE HIS GRATITUDE TO THOSE WHO HAVE CONFERRED

BENEFITS ON HIMSELF AND FAMILY;

THIS WORK,

UNDERTAKEN PARTLY BY THEIR DESIRE,

AND PRINCIPALLY FOR THEIR USE,

IS RESPECTFULLY INSCRIBED,

BY THEIR OBLIGED

AND FAITHFUL SERVANT,

THE AUTHOR.

Gen. Lib.
Gift
Professor William H. Butts
10-14-1935
2 v.

CONTENTS.

ARITHMETIC.

ERRATA.

PAGE

21 Line 3 from the bottom, *for* $\overline{3\times4}$ *read* $\overline{3+4}$.

80 Exam. 45, for *ld.* read *yd.*

Explanation, line 4 from the bottom, for *yards* read *feet.*

140 Exam. 96, *for* 116 *read* 115.

159 Exam. 51, *for* $\frac{13}{20}$ *read* $\frac{10}{21}$.

166 Exam. 83, *dèle* $\frac{2240}{3136}$.

177 Exam. 5, *for* $3\frac{65}{288}$ *read* $2\frac{209}{360}$.

179 Exam. 15, *for* $1\frac{28}{95}$ *read* $1\frac{74}{285}$.

182 Exam. 30, *for* 55*m.* 12″ *read* 58*m.* 32″ $\frac{8}{11}$.

189 Exam. 32, *for* 23*h* *read* 19*h.*

193 Exam. 13, *for* $\frac{1^1}{2\frac{1}{2}}$ *read* $\frac{1}{2\frac{1}{2}}$.

201 Exam. 23, *for* $17\frac{2}{7}$ *read* $17\frac{1}{3}$.

Exam. 24, *for* $4\,\frac{3}{13}$ *read* $4\frac{1}{3}$.

206 Exam. 16, *for* 4*l.* 11*s.* 6*d.* $\frac{33}{40}$ *read* 91*l.* 11*s.* 4*d*$\frac{1}{3}$.

Exam. 19, *for* 8*l.* 19*s.* 6*d.* $\frac{62}{277}$ *read* 9*l.* 11*s.* 6*d.* $\frac{886}{1385}$.

Exam. 20, *for* 1*l.* 2*s.* 5*d.* $\frac{1}{6}$ *read* 1*l.* 13*s.* 7*d.* $\frac{4}{5}$.

235 Exam. 13, *for* $5.4\dot{7}$ *read* $5.\dot{4}\dot{7}$, *for* 3.2 *read* $3.\dot{2}$, and *for* $.12\dot{3}$ *read* $.\dot{1}2\dot{3}$.

236 Art. 253, the demonstration of this rule is by mistake omitted.

240 Exam. 7, *for* 1.14435839, &c. *read* 6.683127, &c.

253 Exam. 20, *for* .295803989, &c. *read* .935414, &c.

257 Exam. 1, in the subtrahend, *for* 576, *read* 5768.

274 Exam. 23, *for* 1162358667 *read* 1162261467.

315 Add at the bottom, as a note on the word ALGEBRA. "Algebra, according to Lucas de Burge, derives its name from the Arabic name of the science, viz. *Alghebra e Almucabala,* or *the art of Restitution and Comparison.*

337 Line 19, for *Leipsigc* read *Leipsic.*

434 Exam. 12, *for* 2*y* *read* 30.

434 Exam. 13, *for* $\frac{y+z}{4}$ *read* $\frac{y+3}{4}$.

Exam. 16, *for* 50 *read* $x+10$.

437 In line 6 of the note *for* $\sqrt{aa-b}$ *read* $\sqrt{\tfrac{1}{4}\,aa-b}$.

442 Exam. 22, *for* $\frac{1}{2}a$ *read* $\frac{1}{2a}$.

443 Exam. 32, for $x-$ read $x=$.

PREFACE.

IN order to ensure success in the cultivation of any branch of learning, it is a matter of prime importance to take care that the first principles and elements be thoroughly understood, and firmly fixed in the memory, by a sufficient number of suitable exercises and examples. This salutary maxim we have the advantage of hearing so frequently repeated, that an inattentive observer might reasonably be led to suppose its truth had obtained universal suffrage; but in this he would be mistaken; for though all seem agreed on the subject, the assent is for the most part merely verbal, and, like our assent to truths of higher importance, has too little influence on the practice. It would be beneficial to learning, and consequently to society, if no instance could be adduced to justify this conclusion; but whoever will take the trouble to examine the plan on which the business of some of our schools is conducted, will find abundant reason to acknowledge its truth—he will find that too little attention is paid to the introductory parts of learning, and that pupils are too frequently hurried on from one subject to another, with a rapidity which does not admit of their fully understanding any thing they pass through. This conduct is both cruel and impolitic; it deprives both the learner and society at large, of the benefits which might be expected from talents properly cultivated. But if part of the blame rest with the preceptor, a much larger share attaches itself to his employers, whose impatience for their children's hasty improvement is too generally productive of this abuse. The

preceptor, indisputably the fittest judge of his own business, and consequently of what methods ought to be pursued for the real benefit and improvement of his scholars, cannot always presume to follow the dictates of his own unbiassed understanding; if he did, he might soon "vaunt and vapour in an empty school."

But one of the greatest impediments to successful teaching, is the undue deference which it is the fashion to pay to juvenile opinion; for although the extravagant doctrines of liberty, asserted by some modern philosophers, as far as they relate to politics, are justly exploded as absurd and impracticable, they still possess a considerable degree of influence on our system of education. No sooner has a young gentleman assumed the neckcloth, than he feels himself invested with a degree of consequence, which, a century or two ago, would have been thought dangerous in such hands, and is allowed a right to offer his opinion with unlimited freedom on every subject. It frequently happens that a father entertains such an extraordinary respect for his son's judgment and penetration, that almost every thing, relating to his future studies, is submitted to his own decision: of course, he determines on that which he expects will be attended with the least difficulty to himself; but as he prefers amusements of his own choosing, none that are proposed will suit him exactly. He objects to grammar because its rules are dry; and if he is obliged to learn them, he is sure that his memory will not retain them—he has not a genius for numbers, his father never had—he would consent to learn Algebra, but he has been informed that the symbols employed mean nothing; how then can the science have any meaning or use? but admitting it to be useful, the operations appear so difficult and complicated, that the advantage of acquiring it cannot be worth the trouble. Geometry, according to his determination (for he is always positive), is of no use to any but

common mechanics, and as he is not intended for one, it can be of no service to him. A pupil of this hopeful description, however he may object to the difficulties of learning, has generally sense enough to see the necessity of pursuing (at least in appearance) some one or more of these studies, in order to secure that respectability to which, in spite of indolence, his pride prompts him to aspire. In doing this, if he employs any effort of mind, it is only in contriving how to evade difficulties of every kind, get through the uninteresting rudiments as hastily as possible, and arrive at those parts which seem to promise more pleasure, or less expense of mental exertion. But in this, ere long, he finds his mistake; for, having obtained his wish, he does not fail to prove an incessant torment to his tutor, whose painful duty it now becomes to teach him the application of those fundamentals which he would never take the pains either to understand or remember, and which nothing can induce him now to resume[a].

The consequences resulting from both the above cases, are in general the same. The pupil, weary of a pursuit which he is at length convinced will yield neither pleasure nor advantage, the moment he is completely at his own disposal, quits it with disgust—the money spent on that part of his education is totally thrown away, or would have been better employed in acts of charity; and, what is far worse, those precious hours, days, months, perhaps years, which

[a] If what M. Rollin says be true, viz. "That it is the end of masters *to habituate their scholars to serious application*; to make them love and value the sciences, and cultivate such a taste as shall make them thirst after the sciences when they are gone from school;" what grief, vexation, and disappointment must that master experience who is unfortunate enough to have in his school half a dozen such pupils as we have described. The best preceptor confined to the tuition of such would be in great danger of soon becoming good for nothing; and indeed, opposition of any kind, from whatever quarter it may arise, if it be sufficiently efficacious to disappoint or subvert the tutor's plans, will have a strong tendency to relax his ardour; it will by degrees bring on an increasing indifference to his duty, and at length reduce him to the state of a mere machine.

constituted the only proper season for his improvement, are irrecoverably lost. When arrived at the state of manhood, he cannot but feel his deficiency, and is sometimes almost half inclined to regret his former obstinate misconduct. Nevertheless, he palliates it with the mild name of juvenile indiscretion; attributes the whole to the ignorance or negligence of his tutors, whose peculiarities (and perhaps their virtues) are the occasional subjects of his merriment; and if he has children, he educates them as nearly as possible after the same plan on which he himself was educated.

These are some of the bad effects which follow from parental authority being misapplied, ineffectually exerted, or not exerted at all; and might easily be avoided, if parents, with due attention to their children's talents, would themselves resolve on the studies to be pursued, and leave the plan and execution entirely to the wisdom and known fidelity of the master; and it is a happy circumstance for learning and mankind, that to the prevailing custom there are many exceptions of this kind. We readily admit, that our ancestors erred, by introducing too much strictness and rigid formality into their mode of instruction; but it is no less certain that the present generation deviate in theirs full as widely towards the opposite extreme, and, from a due comparison of both, it appears that the last error is by far the worst [b]: but as the removal of the cause is not likely to be effected, various contrivances have been resorted to, in order to counteract as much as possible its bad effects: every possible means has been employed to allure the dull, the idle, and the frivolous of every description, to the pursuit of

[b] Dr. Knox delivers his opinion very freely on this subject; let the attentive observer determine how far it is correct. "It is certain" (says the Doctor), "that schools often degenerate with the community, and continue greatly to increase the general depravity, by diffusing it at the most susceptible periods of life. The old scholastic descipline relaxes, habits of idleness and intemperance are contracted, and the scholar often comes from them with the acquisition of effrontery alone, to compensate for his ignorance." *Knox on Education*, p. 31.

knowledge, as well as to assist and promote the progress of real genius and industry, by removing obstacles, and making the way plain and easy. Hence arises the multiplicity of easy introductions, easy grammars, games, &c. which we have in every branch of learning, works which are all useful as far as they go; but it must be remarked, that if they remove difficulties out of the scholar's way, instead of teaching him how to encounter and surmount them, these performances, however they may be patronized and praised, are of but little value. Scientific games, it is allowed, are pleasing and instructive amusements for the nursery; but whatever they may seem to promise, it is making sad game of the sciences, to suppose that these can be acquired by play. I am persuaded that none ever did, or ever will attain to useful or honourable proficiency in any branch of learning, without proportionate labour and application[e].

If what has been said be correct, it will not only account for the great number of easy elementary treatises that has appeared, but will shew that an almost endless variety is absolutely necessary to accommodate the various tastes of learners; it will be a sufficient apology for adding one to the number, as well as for the plan on which it is written.

In the following work, it is proposed to combine more advantages than are to be met with in any single book on the subject, viz. historical, theoretical, and practical knowledge, and to accompany the whole with explanations so exceedingly simple and easy, that it is presumed to be im-

[e] "Nothing can be more absurd" (says the author of Hermes) "than the common notion of instruction; as if science were to be poured into the mind like water into a cistern, that passively waits to receive all that comes. The growth of knowledge resembles the growth of fruit; however external causes may in some degree co-operate, it is the internal vigour and virtue of the tree that must ripen the juices to their just maturity." *Harris.*

[f] "To lead a child to suppose, that he is to do nothing which is not conducive to pleasure, is to give him a degree of levity, and a turn for dissipation, which will certainly prevent his improvement, and may perhaps occasion his ruin." *Knox on Education*, p. 19.

possible that any person of moderate talents will fail to understand them. It supposes the learner to be, in the proper sense of the word, *a beginner*, consequently unacquainted with even the rudiments of science; and from common principles known and acknowledged by all, it proceeds by easy and almost imperceptible gradations, to lead him on (with the aid of *Simson's Euclid* and a Table of Logarithms, both which it explains) to the attainment of a considerable degree of mathematical knowledge, with scarcely any assistance from a master. The work is divided into ten parts, in which the subjects treated of are—Arithmetic, Algebra, Logarithms, Common Geometry, Trigonometry, and the Conic Sections; each preceded by a popular history of its rise and progressive improvements: to which are added, by way of notes, brief memoirs of the principal authors mentioned in the text; some account of their writings, discoveries, improvements, &c. with a variety of useful information of a miscellaneous nature, respecting the Mathematical Sciences.

Part I. begins with an Historical Account of Arithmetic[d], explaining, to a considerable extent, the nature and construction of numbers, and proceeds by laying down in a plain and simple manner, what are usually called the four fundamental rules: next follow in order, Reduction, the Compound Rules, Proportion Direct, Inverse, and Compound; the Rules of Practice, the theory and practice of Fractional Arithmetic, Vulgar, Decimal, and Duodecimal; Involution, Evolution, and Progression, both Arithmetical and Geometrical; the whole demonstrated, exemplified, and explained: and as simplicity and clearness were always the objects aimed

[d] Plato calls Arithmetic and Geometry "The wings of the mathematician;" "Arithmetic" (says M. Ozanam) "may be considered as the mathematician's *right* wing, because without this Geometry would be very imperfect; this justifies the common practice of beginning the Mathematics with the study of Arithmetic."

at, it is hoped no obstacle will be found in the learner's way which may not easily be surmounted. Under these heads, which comprise the whole of Elementary Arithmetic, is given a great number of particular rules and observations, not to be found in any other work, but which are necessary, in order fully to explain the theory, and facilitate the practice of numbers. Besides the examples fully wrought out and explained, several others are introduced under each rule, with their answers only, and a few are given without answers. Part II. contains an Historical Account of Logarithms, the theory and practice of Logarithmical Arithmetic, with numerous examples, problems, and explanations. Part III. contains the History of Algebra, and its fundamental rules; Rules for solving Simple and Quadratic Equations, in which one, two, three, or more unknown quantities are included; and, lastly, a collection of Problems, teaching the application of Simple and Quadratic Equations, in a great variety of ways; the whole accompanied with notes and easy explanations as above. This completes the first volume.

The second volume (Part IV.) begins with Literal Algebra, in which the Problems are analytically investigated, and likewise demonstrated by the method of Synthesis. General conclusions are applied to particular examples, and the methods of converting numeral Problems into general ones; deducing Theorems, Rules, and Corollaries; registering the steps of operations, &c. are laid down and applied in a variety of cases. The doctrine of Ratios, Proportion, Progression, Variable and Dependant Quantities, Interest, Discount, Permutations, Combinations, the Properties of Numbers, &c. are Algebraically investigated, with numerous examples. Part V. explains the nature and theory of Equations in general, their Composition, Depression, Transformation, and Resolution, according to the methods of *Newton*, *Cardan*, *Euler*, *Simson*, *Des Cartes*, and others. Various methods of Approximation as laid down by *Simson*, *Raphson*,

Hutton, *Bernoulli*, &c. the Solution of Exponential Equations, and Problems for exercise. Part VI. explains the nature and method of resolving indeterminate Problems, both simple and Diophantine. Part VII. shews how to convert Fractions and Binomial Surds into Infinite Series, by *Sir Isaac Newton's Binomial Theorem*, and otherwise: how to sum, interpolate, and revert a given Series; to which is added, the Algebraic investigation of Logarithms, with Rules for constructing entire tables of those numbers, both common and hyperbolical. Part VIII. treats of Geometry, viz. its history and use, and describes the nature, construction, and use of Mathematical Instruments, to prepare the learner for the practical application of Geometry: this is followed by an easy logical Introduction to the study of *Euclid's Elements*, considered as a system of demonstration, with Observations on the Definitions, Postulates, and Axioms, and the most remarkable propositions in the first six books, as they stand in *Dr. Simson's Translation*; with Corollaries, explanations of the difficulties that occur, &c. partly original, and partly selected from *Clavius*, *Barrow*, *Saville*, *Austen*, *Ludlam*, *Ingram*, and *Playfair*: to which is subjoined, an Appendix, containing some useful propositions, not in *Euclid*; and an easy system of Practical Geometry and Mensuration, for the purpose of applying *Euclid's* theory to practice. Part IX. contains the theory and practice of Trigonometry, the investigation of Formulæ for the Sines, Tangents, Secants, &c. both natural and artificial; with the description, construction, and use of instruments employed in Altimetry, Surveying, Geography, Navigation, &c.; and, lastly, the Mensuration of inaccessible heights and distances. In Part X. is given the History of the Conic Sections, with the principal and most useful properties of those celebrated curves, deduced by an easy and natural method, accompanied with numerous references to *Euclid*, for the convenience of the learner.

Such is the plan of the work; with respect to its execution,

the Author submits with becoming diffidence to the judgment of the public: he is aware of many imperfections, and is too well acquainted with himself, not to suspect that some errors may have escaped him, of which he is unconscious; but he trusts that none will be found of sufficient importance to mislead the student, or materially impede his progress. If any plea for indulgence could be urged or admitted, it might be truly said, that few books have been composed under more unfavourable circumstances than this; but waving every claim of the kind, the Author only requests it will be remembered, first, that his work is intended for beginners; this will account for some apparent prolixity, especially in the explanations, and for the manner in which some of the rules and operations are accounted for, being rather popular than scientific. Secondly, that as he was not within fifty miles of the press, typographical errors are unavoidable; but from the kind attention of two learned friends at Oxford, and the care taken by the Printers, their number is comparatively small. Thirdly, part of the manuscript was at the press and inaccessible to the Author, while he had the other part under correction, which will explain the cause of two or three unnecessary repetitions, should they be discovered.

It ought to be acknowledged, that in the prosecution of the subjects here treated of, occasional assistance has been derived from the writings of approved authors and commentators; and, in some instances, their methods and observations have been extended, abridged, or otherwise altered, to suit the plan of the Author: this is allowable in works of an elementary nature [e], and is not without its advantage, both to the subject and to the reader.

Two copies of the greater part of Vol. I. having been more than twelve months in boards, were, by way of experiment,

[e] Veneror inventa Sapientiæ, inventoresque adire tanquam multorum hæreditatem, juvat. Mihi ista acquisita, mihi laborata sunt. *Seneca.*

put into the hands of some young persons, not remarkable for their abilities, and who had hitherto experienced considerable difficulties in understanding the treatises of *Walkingame, Bonnycastle*, &c. By the use of these copies only, they have, during the above period, made considerable progress, with very little trouble to themselves or others; having each, on an average, made only about five or six applications per month for assistance, beyond what the book supplies. This fact, which from its nature does not depend on single testimony, is highly gratifying to the Author, as it evinces the usefulness of his work, and, he hopes, will operate as a recommendation; at the same time, he wishes to take no improper advantage of the public, but that it may abide a fair trial; and as he would not willingly expose himself to merited ridicule, by becoming his own panegyrist, it remains only for him to adopt the poet's candid request—

———————— "Si quid novisti rectius istis;
"Candidus imperti: si non, his utere mecum."

The Author is truly sensible of the honour done him by his pupils and others, whose respectable names compose the list of subscribers: he desires, in particular, to express his gratitude to Dr. Macbride and Richard Berens, Esq. LL. D. for their friendly advice and occasional corrections; for their kindness in undertaking the sole management of getting the work printed, and for other favours.

A BRIEF AND GENERAL ACCOUNT

OF THE

MATHEMATICS.

Pure and Mixed Mathematics defined, and their Nature and Uses explained.

THE term MATHEMATICS[f], in its original acceptation, means Learning, Science, or Discipline; but in a more restricted and commonly received sense, it is the science of quantity, which treats of magnitudes, considered either as computable or measurable.

The Mathematics comprehends several branches, each of which ranks as a distinct science: these are arranged under two general heads, viz. Pure Mathematics, and Mixed Mathematics.

PURE MATHEMATICS treats of magnitude generally, simply, and abstractedly; it determines the properties and relations of magnitudes and quantities, considered purely as such, and without relation to any material substance whatever. This class comprehends ARITHMETIC, or the science of Numbers; ANALYSIS, or the science of general calculation; GEOMETRY, or the science of local extension; and MIXED GEOMETRY, in which Arithmetic and Analysis are combined with pure Geometry.

MIXED MATHEMATICS is pure Mathematics applied to Natural Philosophy or Physics; it combines the properties of Body, Motion, &c. as determined by incontestable experiments, with the doctrine of pure quantity; whence by a methodical and demonstrative chain of reasoning, it deduces conclusions as incontrovertibly evident, as those which Pure Mathematics derives from self-evident principles and definitions.

[f] The word *Mathematics* is derived from μάθησις, discipline or science.

The following sciences are comprehended under Mixed Mathematics; viz. 1. MECHANICS, or the science of the equilibrium and motion of solid bodies, treating of the properties and effects of the five mechanical Powers, viz. The Lever, the Axis in Peritrochio, the Pulley, the Inclined Plane, the Wedge, and the Screw; and also of machines of every description compounded of two or more of these. 2. HYDROSTATICS, and HYDRAULICS, comprehending the theory of the nature, gravity, pressure, and equilibrium of fluids; the theory of pumps, syphons, and artificial water-works of every description; to which may be added PNEUMATICS, which treats of the weight, pressure, elasticity, &c. of air and elastic fluids, the air pump, air gun, &c. 3. ASTRONOMY, or the science which treats of the motions, periods, eclipses, distances, magnitudes, and other phenomena of the celestial bodies. 4. OPTICS, or the doctrine of vision, light, and colours, the theory of the eye, the telescope, the microscope, spectacles, and all kinds of reflecting and refracting glasses, to which may be added PERSPECTIVE, or the theory by which visible objects are accurately represented on a plane. 5. ACOUSTICS or PHONICS, which treats of sound, explaining the nature of the ear, of speaking and hearing trumpets, whispering galleries, &c. including MUSIC, or the science of harmony in sounds.

From various combinations of these branches of mixed Mathematics, we derive a great number of additional branches, as SURVEYING, or the art of dividing, delineating, and measuring land. ARCHITECTURE, civil, military, and naval, or the art of planning and building houses, churches, palaces, castles, ships, &c. PYROTECHNIA, or the art of constructing and managing fireworks, gunnery, &c. NAVIGATION, or the art of conducting a ship at sea from one port to another. CHEMISTRY, or the art of decompounding substances, both solid and fluid, by means of fire. ELECTRICITY, or the investigation of certain powers and their effects, as found in amber, sealing wax, glass, tourmalin, &c. and indeed every part of natural philosophy, and every manual art that can be practised, is connected more or less with mixed Mathematics.

The method by which mathematical truths, which before were doubtful, become evident to the understanding, is called DEMONSTRATION: it is this that peculiarly characterizes accurate knowledge, or true science, and distinguishes it from that spe-

cies of knowledge which arises from conjecture, probability, testimony, induction of facts, &c. The latter kinds are called moral evidence, because they are chiefly employed on subjects connected with moral conduct.

On the Difference between Mathematical Demonstration and Moral Evidence.

Demonstration differs from Moral Evidence in the following particulars [g].

1. Demonstration is employed about abstract truths, and the necessary relations of ideas, viz. such as are connected with extension, duration, weight, force, velocity, with whatever else can be accurately expressed by numbers and lines. But Moral Evidence relates to matters of fact, and the constant or variable connections which subsist among things actually existing.

2. They are conducted in a different manner. In demonstration we proceed from known truths to those which were unknown, by steps, each of which is necessarily connected with that which precedes it. In a moral proof, there is no such necessary connection, but it generally consists of a number of independent arguments.

3. In Demonstration it is only necessary to consider one side of the question; for if by Demonstration justly conducted, a proposition be proved true, it is of no consequence what may be urged against it; for whatever is offered as proof on the opposite side, must be a mere fallacy. But in Moral Evidenee, there are frequently cogent arguments on both sides of the question; both sides therefore must be carefully examined, and the assent given to that which is supported by the strongest evidence.

4. The contrary to a demonstrated proposition is not only false, but absurd. But the contrary to a proposition established by Moral Evidence, although false, is not necessarily absurd.

[g] These particulars were taken for the most part from Gambier's *Introduction to the study of Moral Evidence*, *Chap.* 1. A work which I hazard nothing in saying fully merits the encomium which Dr. Johnson has applied to Watts's Improvement of the Mind, viz. "Whoever has the care of instructing others, may be charged with deficience in his duty, if this book is not recommended." While the student is engaged in the study of Euclid's Elements, he should embrace occasional opportunities to read Duncan's or Watts's *Logic*, and these should be succeeded by Watts's *Improvement of the Mind.*

5. In demonstration there is a necessary connection between the successive steps, and hence the ideas compared are immediately perceived to agree or disagree. But in Moral Evidence their agreement or disagreement is only presumed; and that on proofs, which are in their nature fallible; the former therefore produces absolute, but the latter can, at the most, produce only moral certainty.

6. As Demonstration is always accompanied with certainty, rules laid down, which are in all cases capable of being demonstrated, will infallibly lead to truth. But in Moral Evidence, no rules can be given, which will direct us how to form an infallible judgment in any particular case.

7. Demonstration terminates in certainty, which is always absolute, and cannot admit of degrees. But the degrees of moral assent may be indefinitely various, from suspicion up to moral certainty.

8. Demonstration requires no accumulation of evidence; for the truth of a proposition as effectually appears from one proof, properly conducted, and as completely commands our assent, as from many. But Moral Evidence admits of, and frequently requires an accumulation of proofs, and each independent argument in favour of the thing to be proved, increases the weight of evidence, but the whole does not compel the assent.

9. In Demonstration we may reason safely from a conclusion already established; and upon that establish a second, upon these a third, and so on to any length. In Moral Evidence, we can seldom proceed with complete safety beyond the first step: for the second step will be less certain than the first; the third less certain than the second, and so on.

10. All the terms used in a system of Demonstration are previously defined with the greatest accuracy, and are always used in the same sense, so that no dispute can arise, nor any ambiguity have place in their application. But the terms employed in Moral Evidence are not always accurately defined; they are frequently susceptible of very different meanings, and consequently must often lead to uncertainty.

Hence it appears that Demonstration is vastly superior to Moral Evidence. But on the other hand, Moral Evidence is by no means to be lightly esteemed; for although the former is in all cases absolutely conclusive, and the latter not so, yet the

cases to which Demonstration applies, are very few indeed compared with those which we are obliged to believe on Moral Evidence: we act upon the latter in nearly all the affairs of life: "it is frequently the only light afforded us to form our opinion of facts, and to regulate our conduct with respect to them. Without attending to Moral Evidence, we can neither act, nor cease to act; we cannot even subsist without acting upon it, since it cannot be demonstrated that our food will not poison instead of nourish us."

Thus much it was thought necessary to add on this subject, to caution the mathematical student never to attempt the extension of his demonstrative powers to objects beyond their proper sphere. Some have greatly erred in this particular, who, because they could demonstrate a few things, became so unaccountably enamoured with the seducing charms of demonstration, as scarcely to believe any thing on evidence short of mathematical proof: and from this absurd and dangerous scepticism, by an easy and natural transition, have at length terminated their earthly career in the gloomy and fearful character of confirmed atheists.

A farther Account of the Usefulnes of the Mathematics.

It is impossible by any description, to produce in the mind of one unacquainted with the subject, an adequate idea of the usefulness of mathematical learning; he alone who has made considerable progress in the acquisition, will be able in some sort to appreciate its value: it will however be necessary, for the information of the unlearned, to observe, that mathematical knowledge supplies most of the means by which the businesses and affairs of Society are carried on. To mathematical principles the skilful architect necessarily resorts for the means of contriving and executing his plans, and to them every building, from the cottage to the palace, owes its existence. By Mathematics, stately ships, which repel invasion from our coasts, assail with British thunder the fleets and maritime fortresses of our foes, or waft to the alternate shores the produce of every climate, are constructed. By it the mariner shapes his course through the wide and pathless ocean, and the intrepid soldier plans his operations for the honour and protection of our country, and the annoyance of our enemies: versed in mathematical knowledge, the miner contrives his subterraneous excavations, and performs

them with safety and success, procuring metals and minerals for our use, and fuel for our fires. By this the skilful Artist delineates with accuracy, faithful transcripts of Art and Nature; and the Musician extracts harmony from the animal, vegetable, and mineral Creation. Guided and assisted by mathematical reasoning, the ingenious Mechanic contrives and executes the various machines in use among us, and calculates their powers and effects; to mathematical principles almost every tool, or instrument we use is indebted for its existence or perfections, and by them, the extent and value of property of every description, are accurately computed and ascertained.

Hence it appears, that Mathematics, considered only with respect to its immediate application, is indispensably necessary to complete the philosopher [h], the mariner, the soldier, the mechanic, and the artist; but this excellent science, (although fundamental to them) is by no means confined to practical arts and necessary computations [i], it claims an important pre-eminence which renders it worthy the attention of all who aspire to the rank of rational beings. The Mathematics is a system of pure, strict, solid, and conclusive reasoning, calculated to strengthen, extend, and invigorate the powers of the mind [k], to

[h] The Rev. W. Jones very justly remarks, that, "in philosophy, especially under the present state of it, the use of mathematical learning is unquestionable;" and Dr. Watts says, that "the moderns have found a thousand things by applying mathematics to natural philosophy, which the ancients were ignorant of." See *Jones on the use of Math. Learning*, p. 36; and *Watts's Logic*, p. 80.

[i] Jones on mathematical learning.

[k] Dr. Watts observes, that "the greatest clearness of thought and force of reasoning abound in the mathematical sciences;" and he adds, that "if there were nothing valuable in them for the uses of human life, yet the very speculative parts are well worth our study; for by perpetual examples they teach us to conceive with clearness, and to conduct our ideas and propositions in a train of dependence; to reason with strength and demonstration, and to distinguish between truth and falsehood." "The Mathematics" (says Mr. Locke) "is eminently serviceable to strengthen and improve the intellectual faculties, and fit them for every kind of speculation. Would you have a man reason well, you must use him to it betimes, exercise his mind in observing the connection of ideas, and following them in train; nothing does this better than Mathematics, which therefore, I think, should be taught all those who have time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures." "The ancients," says professor Duncan, "who so well understood the manner of forming the mind, always began with the Mathematics—here the understanding is by degrees habituated to truth, contracts in-

guide and assist us in the investigation of truth; applicable to every enquiry where quantity is concerned, to which human reason is competent, and capable of conducting our researches with unerring certainty to the utmost limit to which the powers and faculties of the mind can extend.

From what has been said on the subject, the ingenious student will be able to form his own judgment; he will perceive that this science is of general utility, absolutely necessary to all who are engaged in any learned profession or art, useful in trade, and ought to be studied by the Physician, the Lawyer, the Divine, and by all who have sufficient opportunity.

But it must not hence be inferred, that every one is bound to

sensibly a certain fondness for it, and learns never to yield its assent to any proposition, but where the evidence is sufficient to produce full conviction. For this reason Plato has called mathematical demonstrations *The Cathartics or Purgatives of the Soul*, as being the proper means to cleanse it from error, and restore that natural exercise of its faculties." Lord Bacon represents the Mathematics as eminently adapted to correct the irregularities, and remedy the defects of the mind, equally beneficial to its faculties as athletic exercises are to those of the body, producing strength, vigour, and activity. Of the same opinion is Mr. Harris, who asserts that "every exercise of mind upon theorems of science, like generous and manly exercise of the body, tends to call forth and strengthen nature's original vigour. The nerves of reason are braced by the mere employ, and we become abler actors in the drama of life, whether our part be of the busier, or of the sedater kind." The Rev. W. Jones of Pluckley expresses similar sentiments in nearly the same words; and many other testimonies equally unexceptionable in behalf of mathematical learning might be added; but I shall conclude this long note with part of the Inaugural Oration of Dr. Barrow on his entering the Lucasian professorship of Mathematics at Cambridge in 1663. "The Mathematics," says he, "effectually exercise, not vainly delude, nor vexatiously torment studious minds with obscure subtilties, but plainly demonstrate every thing *within their reach*, draw certain conclusions, instruct by profitable rules, and unfold pleasant questions. These diciplines likewise inure and corroborate the mind to a constant diligence in study; they wholly deliver us from a credulous simplicity, most strongly fortify us against the vanity of scepticism, effectually restrain us from a rash presumption, most easily incline us to a due assent, and perfectly subject us to the government of right reason. While the mind is abstracted and elevated from sensible matter, distinctly views pure forms, conceives the beauty of ideas, and investigates the harmony of proportions, the manners themselves are insensibly corrected and improved, the affections composed and rectified, the fancy calmed and settled, and the understanding raised and excited to more divine contemplations."

aim at becoming a profound mathematician: our principal attention ought to be directed to the duties of the profession for which we are intended, or in which we are engaged, and to the acquirement of the parts of learning immediately connected with it.

Deep and difficult researches in the abstruse regions of science are indeed the business of a few, but they are not necessary to the greater portion of mankind; a general acquaintance with the sciences, provided the elementary branches be well understood, will be abundantly sufficient for the student, who does not intend to make this part of learning his chief study; it will enable him to understand the works on Philosophy, Astronomy, &c. which are usually read, and will be fully adequate to all the purposes of the general scholar.

A few Hints, which it is hoped will prove useful to the mathematical Student.

Our faculties, both mental and corporeal, are talents which our all-wise Creator, to accomplish his beneficent purposes, has committed to us in trust; and the cultivation of them is by no means optional; the duties we owe to God, our neighbour, and ourselves, require their utmost exertion, and strongly imply the necessity of their improvement, and that for the neglect, we shall one day stand accountable. The man possessing an improved understanding is certainly better qualified to benefit society than he who is ignorant; and if the obligation be equally binding on those who have learning, and those who possess the means, but want the inclination to obtain it, how culpable must he be who neglects or refuses instruction! What a pleasing advantage does the devout and serious mind enjoy, which is capable of tracing the Almighty in his magnificent works of creation! which, aided by the light and assistance of science, is enabled to contemplate the universal scale of being through its successive gradations from the minute to the stupendous, and at every step to discover convincing proofs of the power, the majesty, the wisdom, and the unchangeable goodness of God. This is equally the duty and the privilege of rational creatures, and although it ranks no higher than as a branch of natural religion, yet the sincere Christian will feel it of too great importance to be overlooked; it is perhaps the only view in

which he can contemplate sublunary things to advantage, and is fitted to supply one source of comfort, to cheer him under the difficulties and distresses of life.

In concluding this part of the subject, it will be proper to offer a few observations, whereby the student will be the better prepared to decide on the comparative value of the science here recommended.

Philosophy, or the knowledge of men and things, is the end to which learning aspires. The Mathematics (considered with regard to its practical application) is nearly the same to natural philosophy and the arts, as grammar is to language. Neither Mathematics nor grammar can teach its own use, nor be of any service except as introductory to subjects altogether distinct from, and superior to them.

Wisdom is the right use of knowledge, and ensures to us the attainment of the CHIEF GOOD; without wisdom all our proficiency in knowledge, as far as it concerns ourselves, will ultimately be of no advantage. Wisdom includes knowledge and rectitude of mind; knowledge is obtained by learning, especially by the mathematical sciences, which strengthen the mind, improve its faculties, and enlarge its powers in a wonderful manner; but rectitude of mind, without which knowledge is useless or pernicious, is not to be attained by science, or by any methods except those which the Holy Scriptures prescribe; this alone is calculated for the double purpose of stamping a just value on learning, and securing our eternal interests. Compare the longest life with eternity, and it will be found less than a drop of water when compared with the ocean: this should teach us in what proportion our anxiety for *the present* and *the future* ought to be regulated. But there is a happy art by which the interests of time and eternity may be both efficaciously pursued together, without obstructing each other. Let this be the student's chief aim, that, like a truly Christian Philosopher, he may "so pass through things temporal, as finally to lose not the things eternal."

There is one more particular on which (as an instructor of youth, and feeling that I am accountable for the advice I give to a tribunal from whence there is no appeal) it is my duty to offer a few words. We highly value those by whose labours, inventions, or improvements in any useful branch of knowledge, have been

made, and we honour their memory: all this is perfectly right; they are benefactors to mankind, and as such are entitled to our esteem and gratitude: but our applause ought by no means to exceed its just bounds; let it be always remembered that God is the only source of light, knowledge, wisdom, and power;—"the author and giver of every good and perfect gift;"—the GREAT FIRST CAUSE of whatever is calculated to promote the comfort and happiness of man. When any discovery is made, let it be remembered, that it is his good Providence which by latent means brings about the concurrence of events by which it is effected, and therefore to him we ought to consecrate not only our first, but a lasting tribute of praise. But man, who is merely the instrument, acting by means of powers with which (independent of his own consent or agency) he has been previously endued, does not scruple to give and receive that honour which is due only to the Creator.

——————————"Men homage pay to Men,
"Thoughtless beneath whose dreadful eye they bow."

Therefore, while we erect magnificent statues to record the worth of philosophers, statesmen, and heroes, whose lives have been devoted to the service of mankind, and whose actions claim our admiration and gratitude; let us not be forgetful of Him, from whom they derived wisdom to plan, and strength to execute; of Him in whom we all "live, move, and have our being:" who has repeatedly declared himself to be "a jealous God," that "he will not resign his honour to another," and that "those who honour him he will honour; but those who despise him shall be lightly esteemed."

But, notwithstanding these awful sanctions, there are to be found some who boldly attempt to exclude THE ALMIGHTY from his own creation, and others, still more impiously daring, who pretend to prove by arguments drawn from the sciences, which they pervert and misapply, that there is no such Being[1].

[1] Of the former kind may be reckoned those who deny the existence of a superintending and particular Providence; they pretend that the GREAT CREATOR having made and furnished the world, put it in motion, and imposed on it certain laws, has left it entirely to its own care and management. To this class belong all the numerous worshippers of *nature* instead of Nature's God, and those who ascribe all events to *fortune*, *luck*, or *chance*, three imaginary agents which have been aptly called *the Fool's Trinity*. Of the latter kind the refined philosophy of modern times has produced multitudes. Berkeley and

Against these merciless sophists, who (while they attempt to deprive even the wretched of his only consolation, *Hope*) call themselves philanthropists, it is the duty of every friend of religion, virtue, science, and mankind, to make a resolute stand. Those especially, to whom the sacred charge of education is entrusted, should be careful to guard the tender minds of their pupils against the dangerous tenets of such infidel writers, by embracing fit opportunities of exposing the fallacy, inconsistency, and evil tendency of their arguments; by pointing out the nature, object, and limits of demonstration, and of moral evidence; shewing how these differ, and instancing proper cases

Hume (the former in his *Principles of Human Knowledge*, and *Dialogues between Hylas and Philonous*; and the latter in *A Treatise on Human Nature* and his *Essays*) have attempted to prove, that we cannot be certain of our own existence, or that any thing exists, &c. Now the obvious inferences which follow from this and other articles of their creed (which we cannot here enumerate) are such as are destructive of all distinction between virtue and vice, and lead directly to atheism. Some of the inferences which Hume has drawn from his own sceptical theory, are of so blasphemous a tendency, that Dr. Beattie (his great castigator) refuses to commit them to paper. Gibbon is a most dangerous writer on the side of infidelity; under a shew of great candour and acquiescence, the attentive reader may perceive, that he is always aiming a mortal stab at Revelation. Of Voltaire it has been truly said, that he "built God a church, and laughed his word to scorn." Had this elegant writer been content to employ his satirical talents in exposing the knavery of quacks and pretenders of every description, without attempting to undermine and subvert those principles on which alone mankind's best hopes are founded, every honest man would have been obliged to him. D'Alembert, Diderot, Condorcet, La Place, Robinet, and others of the compilers of the French Encyclopedia have done their utmost to make that extensive work a vehicle for disseminating the poison of infidelity. Of Halley, Robins, Thomas Paine, Godwin, Darwin, Walcot, and the writers of some of our magazines and reviews I shall say nothing; it will be necessary merely to inform my youthful readers, that every cavil which these or others have started against the truths of Christianity, has been refuted over and over again; but notwithstanding this, they, or their disciples, are daring enough still to persist, and to call on the world to admire their sagacity and penetration; and for what? for reviving old and worn out quibbles, which have been satisfactorily answered a thousand years ago!

Should any apology be required for introducing this subject here, I have only to say, that I have scrupulously copied the practice of these men; they have instilled the principles of infidelity and atheism by means of their books on science; and I endeavour to oppose them, only by declaring what I firmly believe to be true, and beneficial to society, in mine.

to which they severally apply: and, lastly, by constantly referring the whole, and every part of knowledge, back to its divine original. This the illustrious Newton, with a modesty not less conspicuous than his great talents, felt a pleasure in doing: the same has been the practice of Bacon, Boyle, Nieuwentyt, Ray, Derham, Watts, Jones, Ryland, Adams, and others: in this respect, the labours of these great men confer an honour on the sciences, and a lasting obligation on mankind.

On the Rise and Progress of the Mathematics.

The labours of the early inhabitants of the world, were directed solely to the supply of their immediate wants. Necessity, the fruitful parent of invention, exited all their skill and address, to procure the means of sustenance and defence. These first efforts, rude as they were, and unpromising as they would now appear, rank as the lowest step in the progressive scale of knowledge; by means of a series of accumulated improvements which have successively taken place during the space of nearly six thousand years, they have at length grown into that almost endless variety of useful results, which supply the necessities, the conveniences, and the luxuries of the present age.

The subjects of human knowledge, as far as relate to its inquiries, are either theoretical or practical; the former is called SCIENCE, the latter ART; and since *practice* must have been established long before *theory* could be formed, it plainly follows that every science must have been an art at first; and since the observation and comparison of a great number of facts are necessary to constitute a theory, it is evident, that some ages must have elapsed before any thing like science could have been formed, and it must have been still later before sufficient leisure, information, and inclination would concur, to induce any person to commit these facts, comparisons, and observations to writing.

It might be expected, in an inquisitive and learned age like the present, that to trace the arts and sciences back to their origin, would be no very difficult task: but this is not the case; whoever undertakes to examine the history of former times with this view, will find himself grievously disappointed: he will find that whilst the freaks of tyrants, which have desolated the earth, are recorded with disgusting minuteness, the progress of the human mind in quest of knowledge beneficial to mankind, is

almost totally neglected. Scarcely any particulars connected with early researches of this kind have been noticed by historians; and of the few facts which have been casually transmitted to us, the accounts are so involved in obscurity and contradiction, that very little dependence can be placed on them [m]. In what follows therefore, we shall merely notice the countries where the Mathematics are reported to have been first cultivated, and the principal nations and individuals by which they have been successively improved, and at length transmitted down to us.

It is is generally believed that the mathematical sciences were first cultivated by the Assyrians and Chaldeans, the two most ancient nations on record, at an early period after the flood [n]. Of the nature and extent of these sciences, as they then stood, we are not informed; but it is generally believed that these ancient people were accustomed to make celestial observations in order to understand the seasons, and for the purposes of astrology, a fallacious art for which they appear to have been early distinguished. We likewise gather from circumstances, that Numbering, Measuring, practical Mechanics, and Building, were arts with which they were not unacquainted. From Chaldea the science was transmitted to other neighbouring countries, among which Egypt is celebrated as being for a long time the seat, and chief source of learning. The Egyptian priests, "directed by the laws of their institution to study and collect the secrets of nature, were become the depositaries of all human knowledge;" they were consulted on every difficult subject, not only by their own unlearned countrymen, but the most

[m] Dr. Robertson.

[n] With the state of knowledge among the antediluvians we are almost entirely unacquainted. "Tubal Cain was an instructor of every artificer in brass and iron," and Jubal "was the father of all such as handle the harp and organ:" this is, I believe, all that Moses says on the subject. Josephus affirms, that the sons of Seth were astronomers. Indeed we have reason to conclude from the great length of human life, and the consequent excessive population of the world before the flood, that arts and sciences might have been cultivated by them to a considerable extent. Probably Noah's sons possessed all the learning of that period: if so, it will help to account for the skill in Architecture, Astronomy, &c. which we find displayed within about a century after the flood.

renowned philosophers of all the neighbouring parts flocked to them for instruction [o].

The Phenicians were a trading and flourishing people as early as A. C. 1500, they excelled in learning and manufactures, and to them have been attributed the invention of letters [p], arithmetic, commerce, and navigation [q].

[o] According to the president Goguet, Egypt has the honour of being the first nation which established a public Library; this was contained in one of the buildings forming a part of the magnificent tomb of their king Osymandyas, who lived about the time of the Trojan war, A. C. 1200; over the door of this library was written, Ἰατρεῖον ψυχῆς, *The shop for the Physic of the Soul.* See the *Origines Sacræ*, by Stillingfleet, vol. 1. p. 55. The learned and industrious Sir Walter Ralegh has, from the writings of Diodorus, Diogenes Laertius, Iamblicus, Philo Judæus, Eusebius, &c. collected a summary of the Egyptian learning, as it was in the days of Moses; but he does not pretend to answer for its correctness: "It was divided," says he, "into four parts, viz. Mathematicall, Naturall, Divine, and Morall. In the mathematicall part, which is distinguisht into Geometrie, Astronomie, Arithmetick, and Musick, the ancient Egyptians exceed all others." *Historie of the World*, Part 1. B. 2. Ch. 6.

[p] Lucan thus expresses the current opinion on this subject:

Phœnices primi (Famæ si creditur) ausi
Mansuram rudibus vocem signare figuris.

The Phenician alphabet, consisting of sixteen letters, was carried to Greece by Cadmus, A. C. 1498; to these eight were afterwards added, viz. four by Palamedes, A. C. 1190, and the remaining four by Simonides, A. C. 540. The invention of writing is likewise ascribed (by a writer whom Pliny has mentioned) to Memnon, an Æthiopian king, who flourished during the Trojan war.

[q] According to the following verse of Tibullus,

Prima ratem ventis credere docta Tyros.

The Phenicians passed the Strait of Gibraltar as early as A. C. 1250; and (as Bochart supposes) discovered the *Cassiterides*, or Scilly islands, A. C. 904. According to Herodotus (lib. 4.) they were the first who sailed round Africa; for this purpose they were furnished with ships by Nechus, King of Egypt. Sailing from the Red Sea westward, they doubled the Cape of Good Hope, and after three years spent on the voyage, continued their route to Egypt. Three other attempts of the same kind were made, as we are informed, by the ancients, but only one of these succeeded. If the tradition respecting these voyages was known in Europe, it does not seem to have obtained much credit; for we find, that as late as the fifteenth century, the passage round the south of Africa was deemed impossible, and therefore never attempted. The success of Vasco da Gama, who in 1497 doubled the Cape of Good Hope, was heard of, throughout Europe, with the utmost astonishment.

The Phenicians were almost the only people among whom we find any very early traces of a system of Philosophy. Palestine contained public schools for

Thales and other Grecian philosophers travelled through the eastern countries in quest of knowledge, to civilize and enrich their own: Egypt appears to have been their principal resource; from that country they carried the knowledge of the sciences into Greece about A. C. 600. But, however highly the learning of the Egyptian priests may be esteemed, it does not appear that the Mathematics merited the honourable name of a science, until some time after it had passed from them to the Greeks; in the hands of this diligent and ingenious people, a few detached principles, theories, and observations, were digested into form and consistence, and soon began to assume the appearance of symmetry and beauty: we may therefore consider the Greeks as the inventors of science; for, if it be affirmed that they received the principles and materials from other countries, it must likewise be granted that these were altogether rude and indigested, with scarcely any trait that could entitle them to the appellation of science.

Among the earliest of the Greeks who applied to this subject were Thales, Pythagoras, Cleostratus, Anaximander, Œnopides, Anaxagoras, Euetemon, Meton, Zenodorus, Hippocrates, Plato, &c. and the branches chiefly cultivated by these were Geometry, Astronomy, and Arithmetic. The school of Plato produced many excellent mathematicians, of whom Leodamus improved the analysis of his master, Theætetus wrote Elements, and Archytas first applied Mathematics to practical uses. Eudoxus, according to some, was the first founder of a regular system of Astronomy. Aristotle's works abound with Mathematics; and Theophrastus composed a mathematical history.

The building of Alexandria in Egypt by Alexander the Great, A. C. 332. forms an important epoch in mathematical history. This magnificent city, shortly after the death of its founder, became celebrated for its commerce, and still more so as the seat of learning. Ptolemy Lagus, the immediate successor of Alexander over Egypt, established here the famous Museum, consisting of a society of learned men, maintained at the public expense, and employed in the advancement of philosophy, science, and the liberal arts: he founded, besides, a magnificent

teaching the sciences Kirjath Sepher, mentioned by Joshua, Chap. xv. v. 15, A. C. 1444, denotes *the City of Books or Letters*; which name seems to signify, that the city contained a great number of learned men.

library, which his successors endowed with valuable collections of books, amounting in the whole to 700,000 volumes, and likewise with mathematical and astronomical instruments of every description then known. Here were schools of Astronomy, Physic, Theology, &c. Here were constantly assembled learned men and students from every quarter, who met with great encouragement; and in distant countries, it was considered as a high recommendation to have studied at Alexandria. It is scarcely necessary to observe, that favoured by such an institution the Mathematics was cultivated with ardour, and flourished in an uncommon degree. Among the numerous philosophers furnished by these schools Euclid must not be omitted, A. C. 280; he wrote the Elements of Geometry now in use, to which Aristeus, Isidorus, and Hypsicles added the books on Solids. Philolaus asserted the annual motion of the earth about the sun; and Hicetas of Syracuse taught that the earth has likewise a diurnal motion about her own axis. Archimedes was an excellent Geometer, and the inventer of various machines for raising water, lifting heavy bodies, hurling stones, darts, &c. Apollonius Pergæus has left us a masterpiece on the Conic Sections; Sosigenes instituted the Julian year; Hipparchus wrote on the Chords of Arcs; Theodosius on Spherics; and Vitruvius on Architecture [r].

In the first century after Christ lived Menelaus; he wrote on chords and spherical triangles. Ptolemy, who died A. D. 147, has left us an entire summary of ancient Geography and Astronomy; the latter in a work entitled Μεγάλη Σύνταξις, *the great system;* in which he asserts that the earth is the centre of the universe, an hypothesis since distinguished by the name of The Ptolemaic System. Nicomachus is celebrated for his arithmetical, geometrical, and musical works; Theon for his commentaries on ancient geometricians; and Proclus for his commentaries on Euclid. Serenus wrote on the section of the

[r] The learned and eccentric Cardan ranks Vitruvius among the twelve persons whom he supposes to have excelled all others in force of genius and invention; and he thinks Vitruvius would have been deserving of the first place, if it could be certain that he delivered nothing but his own discoveries. These twelve persons were Euclid, Archimedes, Apollonius Pergæus, Aristotle, Archytas, Vitruvius, Alkindus, Mahomed Ebn Musa, Duns Scotus, Suisset, Galen, and Heber of Spain.

Cylinder; Ctesibius and Hero invented pumps, syphons, and fountains; Pappus has left us six books of mathematical collections; we have commentaries on Archimedes and Apollonius by Eutocius, the friend and disciple of Isidore, a learned architect, who built the church of St. Sophia at Constantinople; and Diophantus was the only Greek author who has left us any work on Algebra: but when he lived is uncertain.

After the division of the Roman Empire, A.D. 364, the eastern portion became the retreat of the sciences. But here, in consequence of the perpetual confusion arising from the rapid progress of vice and profligacy, they were but feebly supported, and at length almost wholly confined to the museum and schools of Alexandria. Here the sciences, although in a manner unprotected, still continued to flourish, until that city fell a prey to the victorious arms of the Arabs. In the year 642 Alexandria was taken, and nearly all the documents of science which the world had ever possessed, perished with its fall. The schools were deserted, the philosophers dispersed, and the numerous volumes, which the munificence and learning of the Ptolemies had accumulated, were consigned to the flames [a]; a few only were spared, not from any regard to their inherent worth, but for the beauty and elegance of their execution, which tempted the avarice of their fierce and barbarous possessors.

Historians are at a loss for language to express their horror at this sad catastrophe; but dreadful as it was, like every other

[a] The city was taken by the Arabs on Friday, in the beginning of the month Al Moharrem, and the twentieth year of the Hejira; after they had besieged it fourteen months, and lost before it 23000 men.

Some time after the surrender, John the grammarian, a learned man of Alexandria, having found means to ingratiate himself with the Arabian General Amru Ebn Al As, begged, that as the books of the library were of no use to the Arabs, he might be permitted to have them. To this request the general replied, that as he had not the power to give them, he would immediately write to the Kalif Omar, his prince, to know his pleasure. He wrote, and received for answer, "If the books you speak of are in all respects agreeable to the Koran, *that* is perfect without them, and there is no occasion for them: but if they are contrary to the Koran, they ought to be destroyed; therefore let them be burned." Omar's cruel mandate was unfortunately obeyed with too scrupulous a punctuality; and the numerous volumes supplied fuel during the space of six months for 4000 baths, which contributed to the health and convenience of that famous city.

afflictive event, even *this* was not without its beneficial consequences; for the dispersion of the learned obliged many of them to seek an asylum in the territories of their conquerors; who by degrees, and notwithstanding the prohibitory clauses in the Koran, acquired, by means of unavoidable intercourse, a taste for those sciences which but a few years before they had used all their endeavours to proscribe. The reader who knows how to set a just value on knowledge, need not be reminded, that a tribute of unfeigned gratitude is due to Almighty Providence for an event equally unexpected, as it has been happy in its consequences to us; an event whereby useful learning was wonderfully (not to say miraculously) preserved, and at length transmitted, not without improvements, to the barbarous nations of Europe.

Under the reign of the Kalif Abu Gaifer Almansor, which began A.D. 754, the sciences had taken root, and began to flourish; they were zealously patronized by his grandson Harun Al Raschid, who himself was well skilled in Astronomy and Mechanics. No fewer than eleven Kalifs of the family of Al Abbas are mentioned as cultivators or patrons of science; but not one of all the Arabian princes cultivated the sciences with so much ardour and success as Al Mamon, who ascended the Moslem throne A.D. 813. The virtues and attainments of this excellent prince, and his zeal in the pursuit, encouragement, and diffusion of knowledge, would have done honour to a more enlightened age and nation. On the revival of learning among the Arabs, their first care was to procure the works of the best Greek writers on Arithmetic, Geometry, Trigonometry, Mechanics, Natural Philosophy, and particularly Astronomy; they translated into the Arabian language, Euclid's Elements, Archimedes on the Sphere and Cylinder, with other parts of his works, the Trigonometry of Menelaus, the Conics of Apollonius, Ptolemy's Almagest, Aristotle's Analytics, &c. and enriched their translations with various observations, commentaries, and improvements. The principal Arabian astronomers and mathematicians of this era were Alkindus, Habesh al Merwazi of Baghdad, Ebn Musa al Kowarazmi, Ebn Sahel, Ebn Batrick, Alfragan surnamed *the Calculator*, Ebn Thebit, Musa Ebn Shaker, Abu Gaifar Ebn Musa, Achmed Ebn Musa, Al Hazen, Albategnius, surnamed *the Arabian Ptolemy*, Honain Ebn Al Ebadi, Ishak Ebn Honain, &c.: by the labours

of these and many others, the sciences, which had been in a manner lost, were recovered, improved, and preserved; and from their country at length transmitted into the western nations of Europe [t]. The means by which the latter was effected arose principally from the inroads and final settlement of the Moors or Saracens in Spain, A.D. 713, into which country they carried the sciences; and the necessary intercourse between them and the natives, and the occasional visits of inquisitive foreigners terminated in the gradual diffusion of mathematical learning through the neighbouring countries. From the Arabs the Europeans received translations of several Greek treatises on the sciences, before it was known here that the originals were in existence. Euclid's Elements were first translated from the Arabic into Latin, at a time when no Greek copy was to be found [u].

The Europeans manifested, at first, considerable aversion to the sciences [x]; but by the labours and address of a few learned

[t] The diffusion of science is to be ascribed partly to the Jewish people, whose extensive commerce with almost every part of the world was favourable to this purpose. Historians mention the names of several learned men of the Hebrew nation, among whom are Rabba Judah, Isaac Ebn Baruch, who read lectures on the Mathematics; Beren Al Pherec, Rabbi Abi, Judah de Toledo, Ebn Ragel; Alquibuts de Toledo, Ebn Musio, Mahommed de Savellia, Joseph Ebn Hali, and Jacob Abvena, were skilful astronomers: Ebn Esra was one of the most learned men of the age, (A. D. 1170;) he excelled as an astronomer, philosopher, physician, poet, and critic, and wrote on Geometry, Algebra, Arithmetic, Logic, Astronomy, Astrology, &c. In every country where learning was beginning to be cultivated, the Jews seem to have been among the first of those who were employed as teachers, especially in Britain; probably they were the only persons at all qualified to undertake the arduous and (at that time) unthankful employment. See *Dr. Henry's History of Britain, vol.* iv. *p.* 168.

[u] It is stated in some of our public prints, that Dorville (on Chariton, p. 49, 50) says he was in possession of the works of Euclid in manuscript, dated A. D. 995, which manuscript is now lost: the oldest manuscript at present extant is part of the works of Plato, dated A. D. 996.

[x] It is said that learning, such as it was, flourished in Britain from the end of the first century after Christ to the middle of the fourth, when it began to decline. During the sixth century, this island was a continued scene of war, confusion, and misery; very few paid any attention to learning, and they were despised and insulted on account of it. Books were so extremely scarce, that none but kings, bishops, and abbots, could afford to purchase them: King Alfred, in 690, gave eight hides of land to Benedict Biscop, Abbot of Weremouth, for one single volume of cosmography. There were no schools at this period, except in kings' palaces, bishops' seats, or monasteries; hence the little

persons in Italy, Germany, France, and England, knowledge began to make a perceptible progress. Pope Sylvester II. acquired Arithmetic from the Arabs, A.D. 960; Alphonsus II. King of Castile, founded a College for the advancement of Astronomy, and placed it under the direction of some learned Arabs; Leonard de Pisa was skilled in Algebra, according to M. Cossali, as early as 1202; Jordanus Nemorarius wrote on Arithmetic, Geometry, and the Planisphere, A.D. 1230; and about the same time Johannes de Sacro-bosco, an Englishman, was professor of Mathematics at Paris, and wrote on Arithmetic, the Sphere, the Calendar, and the Astrolabe. Twenty years after Campanus de Novara wrote on the Sphere, Theories of the Planets, &c. and translated Euclid's Elements; and Gerard of Cremona translated the works of Aristotle, the Almagest of Ptolemy, with Geber's Commentary, Alhazen's Treatise on Twi-

learning then in vogue was necessarily confined to princes, priests, and a few of the chief nobility. The eighth century was more dark, barbarous, and ignorant than any preceding; many of the priests could not even read. The ninth century was little better; but it produced a few learned men, as Alfred, Aldhelm, Bede, Egbert, and Alcuinus. The tenth century has been called *the unhappy age*, which "for its barbarism and wickedness" (says Baronius) "may be called *the age of iron*; for its dulness and stupidity, *the age of lead*; and for its blindness and ignorance, *the age of darkness*." The little learning of the eleventh and twelfth centuries (although patronized by princes and great men) was chiefly confined to the monks; it consisted of Metaphysics, Natural Philosophy, Law, Medicine, School Divinity, Geometry, Astronomy, (as these sciences then stood) and Astrology. Ingulphus, Lanfranc, Anselm, Pullus, Eadmerus, William of Malmsbury, Simon of Durham, Matthew Paris, Roger Hoveden, Benedict Abbas, Peter of Blois, and John of Salisbury, were among the most distinguished scholars of this age. The thirteenth and fourteenth centuries are chiefly remarkable for the theological disputes of the schoolmen, and for the vain and ridiculous pursuits of astrologers, magicians, and alchymists, which abounded every where. The fifteenth century (as we have observed above) witnessed the dawnings of science. These dark and barbarous times include that space which is usually denominated in history, *the middle ages*. Those who aspired to the rank of philosophers, in these ages of ignorance, endeavoured to persuade themselves and others, that by consulting the various configurations and positions of the planets, they could determine the future destinies of kingdoms, states, and individuals; they laboured with incessant assiduity to find *the philosopher's stone*, or a composition whereby it was pretended that base metals might be changed into gold; or in equally vain and foolish attempts to discover *the Panacea* or universal remedy, which they supposed would cure every disease and prolong life, if not wholly prevent the approach of death.

light, &c. into Latin; he likewise wrote a work on the planets. In 1260 Thomas Peckam, Archbishop of Canterbury, and Vitellio, a Pole, wrote treatises on Optics, as did Albertus Magnus on Arithmetic, Geometry, Astronomy, and Mechanics: at this period flourished the famous Roger Bacon, who possessed an extensive and accurate knowledge of the sciences rarely to be met with in those barbarous times. The invention of Spectacles, by Alexander de Spina of Pisa, a Jacobin friar, took place about the same date; and that of the Mariner's Compass in 1302. Peter d'Apono wrote on the Astrolabe; and Ascoli, professor of Mathematics at Bologna, composed a commentary on the Sphere of Sacro-bosco; both these were considered as heretics and sorcerers; in consequence of which the former was burnt in effigy, and the latter in person, at Bologna, A. D. 1328. A few other mathematicians flourished in the fourteenth century; as John de Muris, Nicholas d'Oresme, Suisset, John de Lignieres, Bradwarden, &c. But these dark ages are principally famed for the schoolmen, a class which comprised all the learned men of those times; in their hands the Logic of Aristotle became an engine for solving all manner of doubts and difficulties; knotty questions, frequently on the most trivial subjects, in some cases indecent, and in others profane, furnished matter for their almost endless disputes, which were urged with vehemence and acrimony, not so much to discover truth as to obtain victory; indeed the sole tendency of their labours was to obscure truth, and involve the human mind in the grossest ignorance: however, notwithstanding this clouded and inactive state of knowledge, two inventions in mechanics, during this period, deserve our notice, viz. a machine for grinding rags for the purpose of making paper, by Ulman Strame, of Nuremberg; and wheel-work clocks, both fixed and portable.

The fifteenth century, which may be considered as the dawn of science, produced many able mathematicians, among whom may be mentioned John Gmunden, Dailli, George of Trebizonde, Cardinals Bessarian and Cusa; Purbach and Regiomontanus, the two great restorers of Astronomy; Waltherus, Lefevre, Novera, Bianchini, Angelo, Ferdinand of Cordova, Henry Duke of Visco, and Lucas de Burgo, the introducer of Algebra into Europe. This century is famous for the invention of Printing, by Faust of Strasburg, in 1440, and also for the first and grand

applications of the theory of the Loadstone, and of mathematical knowledge to the useful purposes of navigation and commerce, namely, in the voyages of Diaz, Vasco da Gama, and Columbus: the first reached the Cape of Good Hope in 1486, the second doubled it in 1492, and the same year Columbus crossed the Atlantic, and discovered the West Indies.

The sixteenth century is the era of the complete revival of mathematical learning, and of important discoveries and improvements in several of its branches. Algebra is indebted in this respect to the labours of Carden, Ferrei, Tartalea, Ferrari, Bombelli, Maurolycus, Scheubelius, Sturmius, Recorde, Stifelius, Clavius, and Vieta whose improvements were great and valuable. Tartalea, Commandine, Durer, Nonius, Ubaldi, Saville, Ramus, and Vieta excelled in Geometry; Copernicus revived the Pythagorean system of the universe; Tycho Brahe was called *The great Observer*; Kepler was the creator of true physical Astronomy; Schonerus, Fracastorius and William prince of Hesse Cassel were diligent observers of the heavenly bodies; and Aloysius Lilius, an astronomer of Verona, was the person whose plan was adopted by Pope Gregory XIII. for reforming the calendar[y]. John Baptista Porta invented the Camera Obscura; and Maurolycus was a considerable writer on Optics.

The numerous inventions and discoveries which took place in the seventeenth century, advanced the theory and practice of mathematical learning to a pitch until then unknown, and far exceeded the most sanguine expectations or hopes of any preceding age. One of the most noble and useful of these was the invention of the Telescope, which is ascribed by some to John Lippersheim, in 1605; by others to Zachary Jansen; and again by others to James Metius; but it was claimed by Fontana. Kepler first explained this useful instrument; and it received various improvements from Galileo, Reive, Borelli, Hartsoeker, Cox, Campani, Hevelius, Scheiner, Reita, Gregory, Huygens,

[y] The method now in use of computing from the birth of Christ was instituted by Rabbi Samuel, Rector of the Jewish School at Sora in Mesopotamia, probably about the year 250. It was first used in the West, A.D. 527, by Dionysius Exiguus, by birth a Scythian, and at that time a Roman abbot. Venerable Bede employed it in his writings: the recommendation it thereby obtained, occasioned it to be brought into common use, and the great convenience of this epoch has caused it to be retained ever since.

Newton, Caleb Smith, Dollond, Ramsden, &c. Drebell, a Dutchman, is said to have invented the microscope about 1621; but Fontana assumes the honour of the discovery, which he says took place in 1618. Torricellius invented the Barometer; and the Thermometer has been ascribed to different persons, as Galileo, Father Paul, Sanctorio, and Cornelius Drebell of Alkmaer. Antonio de Dominis first explained the phenomenon of the rainbow; as did Wellebrord Snellius the laws of refraction: Columbus first observed the variation of the magnetic needle; Edward Wright discovered the true method of dividing the meridian line, and of constructing the charts usually ascribed to Mercator. Napier was the first who published a system of Logarithms, viz. in 1614, which numbers were greatly improved by Briggs. Stevinus of Bruges was the inventor of Decimal Arithmetic, 1610. Harriot was the father of modern Algebra, 1631. Geometry and Analysis are indebted to Roberval, Cavalerius, Comiers, l'Hôpital, Leibnitz, Mercator, Pascal, Wren, Sauveur, Parent, Barrow, and Wallis, for several new, useful, and interesting theories. Gassendi, Kepler, Picard, Hevelius, Flamstead, Horrox, Recciolus, Hooke, Longomontanus, Kircher, Bayer, and Galileo, were eminent in Astronomy. Des Cartes excelled in Geometry and Algebra, and was the inventor of the *Cartesian Philosophy*, depending on the absurd theory of the vortices, the foundation of which existed only in the imagination of the author. Sir Isaac Newton excelled in almost every branch of knowledge, and appears to have been the highly honoured instrument, designed by Providence to dispel the mists of error which had hitherto inveloped the human mind; his discoveries and improvements in Analysis are numerous and valuable; his Doctrine of Fluxions, in addition to its extensive application, is a masterpiece of ingenuity; he first established the true theory of Light and Colours in his excellent work on Optics: he confirmed the system of Copernicus; and by his discovery of the universality of the principle of gravitation, and his newly invented Analysis, he explained and demonstrated the laws by which that system is regulated. Bacon and Boyle were among the first who taught philosophers to reason from experiment and observation, and to emancipate the human mind from that slavery to Hypothesis, in which it had been triumphantly detained for several ages by the

schoolmen [*]. During this century several Institutions for the joint purposes of cultivating, extending, and registering every part of science were formed in different parts of Europe, viz. at Soissons, Beaujolois, Nismes, Angers, Bologna, Florence, Naples, Verona, Brescia, and Padua. The Royal Society at London was founded in 1660; and the Academie des Sciences at Paris in 1666; the Observatory at Paris was built in 1672; that of Flamstead House, at Greenwich, in 1676.

The recent discoveries and improvements by Vieta, Des Cartes, Harriot, Newton, Leibnitz, and others, opened a new and extensive field for the exercise of talent, in every department of science. The Newtonian Analysis has been applied with equal zeal and success to some of the most difficult and interesting problems in Mechanics, Astronomy, &c. the solution

[*] The learned men who flourished between the time of the Conquest and the revival of learning are usually denominated *schoolmen*, though some writers place them within narrower limits. The schoolmen had the vanity to pretend to account for every thing; they explained the Phenomena of Nature by Hypotheses, instead of facts deduced from experiment and observation; their hypotheses were always conjectural, and very frequently improbable and false, consequently their reasonings and conclusions must have been injurious to the progress of sound knowledge. They substituted hard and unintelligible words and phrases, for causes which they did not at all comprehend, in order to conceal their ignorance; and wished mankind to believe, that by referring to these they had explained the nature of things. Logical arguments were their grand resource, and the defence and support of favourite hypotheses, their chief employ. Their disputations were carried on with a view to obtain victory rather than for the discovery of truth. They infected every subject with their jargon; Law, Physic, Divinity, and Science abound with their sophisticated phraseology, consisting of little else than grave and pedantic displays of ostentatious trifling; especially the books on those subjects written about three centuries ago, which on this account the modern reader will be at considerable loss to understand. The logic of Aristotle, as the grand engine of the schoolmen, has met with indiscriminate censure from a great number of later writers; but I think without justice, as it is not the science, but its misapplication that deserves blame. Bacon, Boyle, Barrow, Locke, and other reformers of science have made great use of the Aristotlean logic in their discoveries, and the only difference is, that the reasonings of these were always founded on truth or (where that could not be obtained) strong probability, and had the discovery of truth for their object; while those of the schoolmen were too often founded on vague or improbable hypotheses, and terminated in procuring their authors undeserved renown, but made mankind neither wiser nor better.

of which had been hitherto sought for by other methods in vain.

The following are the names of some of those who since the commencement of the eighteenth century, have excelled in this and other branches, viz. Mad. Agnesi, D'Alembert, Atwood, De Billy, James, John, and Daniel Bernoulli, Bezout, Borda, Birch, Batten, Browne, Le Bas, Bossut, Barlow, Bonnycastle, Bridge, Cousin, Courtivron, Cotes, Colson, Clairaut, Cramer, Condorcet, Craig, C. and G. Cooke, Christie, Demoivre, Dalby, Dealtry, Dodson, Euler, Emerson, Fontaine, Facio, Fagnanus, Frend, Farish, La Grange, Guisnée, Glenie, Olinthus Gregory, L'Hôpital, La Hire, Hayes, Hornbuckle, Hermann, Hutton, Hustler, Hellings, Jacquier, Jones, Kirkby, Kelly, De Lagni, Landen, Littledale, Manfredi, Monmort, Maclaurin, Montucla, Maseres, Milner, Nicole, D'Omerique, Ozanam, Pemberton, Prestet, Pingré, Peacock, Riccati, Reyneau, Robertson, Rigaud, Sterling, Saunderson, Le Sieur, Saurin, Simson, T. Simpson, Sowerby, B. Taylor, M. Taylor, Turner, Viviani, Varignon, Vince, Waring Wolfius, Watson, Woodhouse, Wood, &c. Astronomy has been cultivated during the same space by many learned men, among which the following are some of the principal, viz. Adams, Bradley, Bouguer, Bailli, Bulkley, the Cassinis, La Caille, Ferguson, Halley, Harding, Herschel, Juan, Koenig, Keill, La Lande, Long, Lax, Maupertuis, Mayer, Maskelyne, Olbers, Pound, Smith, Wolloston, &c. Some of these, with many others, excelled in various branches of the Mathematics, besides those we have ascribed to them; to particularize their inventions, improvements, and excellencies, with just discrimination, would far exceed our prescribed limits; and fully to understand them, recourse must be had to a great variety of modern treatises on every branch of mathematical science.

Thus we have endeavoured to shew, in a brief and general manner, the nature, great importance, and use of mathematical knowledge; and to point out a few of the leading facts in its history. The reader need not be informed, that by the improvements and discoveries in science, which have taken place during the three last centuries, and the application of mathematical and physical knowledge to Civil Polity, the Arts, Commerce, and Agriculture, the present generation enjoys advantages superior beyond comparison to those possessed by former

ages. Our own country was, but a few centuries ago, an overgrown wilderness, a prey to the wildest superstition, and scarcely supplied the bare necessaries of life to its scanty and savage inhabitants. Now, the conveniences and luxuries of every kingdom in the world are poured in, and added to the produce of our own, constituting a rich abundance for the supply of every want, and the gratification of almost every wish; it follows then, that our obligations to Providence are proportionably greater than those of former ages. If the ox knows his owner, and the ass his master's crib, let us not, more stupid and ungrateful than they, while we live in the enjoyment of infinitely superior benefits, be less dutiful in our attachment, nor overlook the kind hand which supplies them. Possessing more ample means than our ancestors possessed, it is incumbent on us to improve our advantages, by the strict and faithful observance of every religious, moral, and social duty.

AN

EASY INTRODUCTION

TO THE

MATHEMATICS, &c.

PART I.

ARITHMETIC.

HISTORICAL INTRODUCTION.

Numerorum notitia cuicunque primis saltem literis eruditio necessaria est.
QUINTILIAN.

ARITHMETIC[a], or the science of Numbers, is justly considered as the basis of all the other mathematical sciences; and therefore a sufficient acquaintance with its principles and elementary rules ought to be acquired before any of the other branches are attempted.

Arithmetic holds a distinguished rank among the mathematical sciences; it even surpasses them all in usefulness: its universal application to the common concerns of life renders it a part of knowledge not merely desirable, but necessary to every one who wishes to be serviceable to society, to manage his own private affairs well, and to guard against fraud and imposition.

Nothing satisfactory can be offered respecting the origin and invention of Arithmetic; like almost every

[a] The word is derived from the Greek ἀριθμος, *number*, and μετρεω, *to measure*.

other useful art, its beginning must have been extremely rude and simple, the fruit of pure necessity, and it must have originated in the first ages of the world, when men began to form societies; for it is not easy to conceive how social intercourse could have been maintained, differences and disputes adjusted, bargains made, and trafficking carried on, without the necessary aid of computation.

Shortly after the dispersion of mankind, the sciences were carried by the descendants of Shem into Chaldæa and the East; in these countries Arithmetic was cultivated probably at an earlier period than in any other. The Phœnicians[b], who were descended from Canaan, the son of Ham, and settled on the eastern coast of the Mediterranean sea, were the people who first of any addicted themselves to commerce, to which they made navigation subservient; and as they must have practised Arithmetic to a great extent in their numerous mercantile transactions, succeeding nations have ascribed to them the invention.

Josephus[c] informs us, that Abraham, having acquired

[b] The Phœnicians inhabited the sea-coast, extending, according to Ptolemy, from the river Eleutherus on the north, to Pelusium on the south. They are called in the sacred writings *Canaanites*, and are remarkable for the series of awful calamities and judgments which a long and uninterrupted course of the most abandoned profligacy had brought upon them. They were, during the captivity of the Israelites in Egypt, (from 1635 to 1491, A. C.) considered as a great and powerful people. Their mercantile spirit and excessive riches are mentioned by the prophets Isaiah and Ezekiel, both of whom denounce the impending judgments of the Almighty on their pride and obduracy. Profane authors speak of their great industry, and represent them as the inventors of letters, arithmetic, commerce, navigation, and almost every thing that is useful.

[c] Flavius Josephus was born at Jerusalem A. D. 37, and died A. D. 93; he was equally great as an historian and an orator, as is witnessed by his "History of the Antiquities and Wars of the Jews." His "Discourse on the Martyrdom of the Maccabees" is a masterpiece of eloquence; he is called by St. Jerome, "The Livy of the Greeks."

a knowledge of Arithmetic in the East, was the first who instructed the Egyptians in the art[c]. By the Egyptian priests Arithmetic was cultivated with ardour, and constituted no inconsiderable part of their theology and philosophy. The Grecian philosophers, who travelled into the East in quest of knowledge, transmitted this science from Egypt into Greece, where it must (in common with the other sciences) have received considerable improvements; among which the invention of the Multiplication Table is ascribed to Pythagoras[d], and a method of determining the Prime Numbers to Eratosthenes[e].

[c] Joseph. Antiq. b. i. c. 8. Abraham was a native of Ur in Chaldæa, from whence he was driven by a famine into Egypt. If the account given by Josephus be true, we are sure that Arithmetic must have been known and practised by the Chaldæans about the time of their first settling in that country.

[d] "The combinations of numbers constituted one of the principal objects of his researches; and all antiquity testifies that he carried them to the highest degree."...... "He attached several mysterious virtues to numbers, and swore by nothing but the number *four*, which was to him the number of numbers. In the number *three* likewise he discovered various marvellous properties; and said, that a man perfectly skilled in Arithmetic possessed the sovereign good." It is supposed by some, that these expressions, and others of a like tendency ascribed to the ancient philosophers, are not to be understood literally, but that they have a figurative and hidden meaning unknown to us.

[e] Prime Numbers are such as cannot be divided by any number greater than unity without a remainder; the rest are called *composite*. The ingenious method alluded to above was called, "*The Sieve of Eratosthenes*:" for some account of it, see Dr. Horsley's paper in the Philosophical Transactions, vol. 62. p. 327. Eratosthenes was a native of Cyrene, a city of Lybia; he was the second person entrusted with the care of the Alexandrian library: grammar, poetry, philosophy, and mathematics, were the subjects that engaged his affections, especially the latter. He measured the obliquity of the ecliptic, making it only about $20\frac{1}{2}$ degrees; he also measured a degree of the meridian, and thence determined with tolerable accuracy the circumference of the earth. The invention of the armillary sphere is ascribed to him; and his consummate skill acquired him the names of *The second Plato—The Cosmographer and Geometer of the World*, &c. He starved himself to death, A. C. 194, in the 82d year of his age. Of his compositions a few fragments only remain.

The Hebrews, Greeks, and Romans represented numbers by the letters of the alphabet peculiar to each nation. The most simple method of notation among the Greeks was, by making their 24 letters represent each a number in order, from 1 to 24; this method may be seen by referring to Homer's Iliad, or Xenophon's Cyropædia, where it is employed in numbering the books: higher numbers were represented by small letters pointed underneath, by the capitals, and by inclosing the capitals with the Greek Π, &c.

From Greece Arithmetic passed to the Romans, who do not seem to have made any improvement in the science; they merely adapted the letters of their alphabet to the numbers received from their masters. Their method, which is still employed in denoting dates, chapters, and sections in books, ought to be understood by every one, and is as follows. I stands for *one*, V *five*, X *ten*, L *fifty*, C *one hundred*, D *five hundred*, M *one thousand*[f]; these seven letters, differently placed or marked, were made to express all numbers. As often as any character is repeated, so many times its value is repeated; thus, II represents *two*, III *three*, XX *twenty*, XXX *thirty*, CC *two hundred*, MM *two thousand*. A less character placed on the left of a greater diminishes its value; thus, IV denotes *four*, IX *nine*, XL *forty*, XC *ninety*. A less character to the right of a greater increases its value; thus, VI denotes *six*, VII *seven*, XI *eleven*, LX *sixty*,

[f] The derivation of these numerals is thus given by some: I, denoting *initium*, the beginning, was considered as the only fit representative of the first number, or *one*. V, (the ancient U,) being the fifth vowel, was with propriety put for *five*. X, being made up of two V's, represented two fives, or *ten*. C, *centum*, or *one hundred*. M, *mille*, or *one thousand*. L, being the half of the old C, which was square, was put for half a hundred, or *fifty*. D, *dimidium mille*, or half a thousand, *five hundred*. The D was frequently written IƆ, and the M, CIƆ; hence these latter marks are sometimes put for 500 and 1000 respectively.

CXX *one hundred and twenty*, DX *five hundred and ten*, DCC *seven hundred*, M,DCCCC,XC,IX *one thousand nine hundred and ninety-nine.* In some ancient books, records, and inscriptions, and on antique coins and medals, we meet with the C inverted; thus, IↃ denotes *five hundred:* every Ↄ added increases it tenfold; thus, IↃↃ denotes *five thousand;* CIↃ stands for *one thousand*, and a C and Ↄ added at the ends increase its value tenfold; thus, CCIↃↃ denotes *ten thousand*, CCCIↃↃↃ *one hundred thousand*, CCCCIↃↃↃↃ *one million*; a line over any number increases its value a thousand-fold; thus, $\overline{\text{VIII}}$ denotes *eight thousand*, $\overline{\text{X}}$ *ten thousand*, $\overline{\text{LXXX}}$ *eighty thousand*, $\overline{\text{CC}}$ *two hundred thousand*, $\overline{\text{MMM}}$ *three million*, &c.

We have not the means of tracing the progressive improvements of Arithmetic among the ancients; judging from their works, (which however are not always to be depended on[a],) there is reason to suppose that the science advanced. Beside Addition, Subtraction, Multiplication, and Division, the ancients possessed methods of extracting the Square and Cube Roots; they were acquainted with the theory of Proportions; Arithmetical and Geometrical Progression; and in general with the combinations of numbers, the reduction of ratios to their simplest form, &c.

The ancient methods of notation were, however, but ill adapted to the practical operations of Arithmetic; and hence it is that the art, with respect to its practical part, must have made but slow progress. The destruction of

[a] Although the greater part of heathen antiquity has descended to us through the hands of the Greeks, yet their evidence must be received with caution, particularly that of the Helladians; they were a bigotted people, highly prejudiced in their own favour. There surely was never any nation so incurious and indifferent about the truth. *Bryant's Analysis*, vol. i. p. 143. 155.

the famous Alexandrian library, A. D. 642, has left us no particular treatise on the subject; we have, however, some of the most plain and useful properties of numbers in the seventh, eighth, ninth, and tenth books of Euclid's Elements, A. C. 280, and in the Arenarius of Archimedes, A. C. 220; there is likewise the Commentary of Eutocius on Archimedes' Treatise of the Circle, some fragments of Pappus[h], A. D. 400. The writings of Nicomachus, A. D. 100, which were published at Paris in 1538, and the treatise of Boethius[i], written at Rome in the sixth century, give us no very favourable idea of the ancient Arithmetic, which seems to have consisted principally of dry and tedious distinctions and divisions of numbers; so that on the whole, the acquisition of any considerable degree of knowledge in this most useful branch must have been attended with almost insurmountable difficulties[k].

[h] Pappus was an eminent mathematician and philosopher of Alexandria; he lived in the fourth century after Christ; the greater part of his valuable writings are lost. His *Mathematical Collections*, in eight books, except the first, and part of the second, are still extant; parts of these have been published by the following authors, viz. Commandine, in a Latin translation with a commentary, 1558; Mersenne, 1664; Meibomius, 1655; Wallis, 1688; David Gregory, 1703; and Dr. Halley, 1706; also Dr. Hutton has given a brief analysis of these books in his Mathematical Dictionary, p. 187, 188. vol. ii.

[i] Boethius was a celebrated Roman; he was put to death, A. D. 525, by Theodoric, king of the Ostrogoths, on suspicion of a conspiracy. During his confinement he wrote that excellent work *De consolatione Philosophiæ*. The best editions of his works are that of Hagenau, 4to, 1491, and that of Leyden, *cum notis variorum*, 1671.

[k] Aldhelm, bishop of Shireburn, and one of the most learned men of the age, who flourished in the time of the Saxon Heptarchy, A. D. 700, complains bitterly of the difficulties he met with in learning Arithmetic, as almost surpassing the powers of the human mind. He thus writes to his friend Hedda, bishop of Winchester. "What shall I say of Arithmetic, whose long and intricate calculations are sufficient to overwhelm the mind, and throw it into despair? All the labour of my former studies, by which I made myself a complete master of several sciences, was trifling in comparison of what this cost me." *Anglia Sacra*, t. ii. p. 6, 7. *quoted by Dr. Henry.*

Psellus, who lived in the ninth century, wrote a compendium of the ancient Arithmetic in Greek, which was published by Xylander, A. D. 1556, in Latin; and a similar work was written shortly after in the same language by Iodocus Willichius. These works are at present objects rather of learned curiosity than use; few persons will take the trouble to understand them.

The Arabs, who had shewn themselves the most inveterate enemies of learning, by a revolution of sentiments not uncommon, became its most zealous supporters. From them Arithmetic received some of its most useful improvements; among which the method of notation at present in use may be considered as the chief. It does not appear, however, that the Arabians ever laid claim to the invention; they refer us to the Indians; and hence the figures employed in our calculations are sometimes called Indian characters.

The Arabs were in possession of the Indian method of notation probably for the space of three centuries before the Europeans knew any thing of the matter. The latter were involved in the darkest ignorance, which the genius and learning of the few great men this age of blindness produced were unable to dispel, and which served only to render that mental darkness visible in all its horrors.

Among the few illustrious characters which appeared at this period, Gerbert[1] deserves the first place. This

[1] Gerbert was born of mean parents, but it is uncertain in what year. Having spent several years among the Saracens at Corduba, during which he industriously collected all that was valuable of their Geometry, Astronomy, and Arithmetic, he returned to France in 970, where he was caressed by the wiser part of his countrymen; but the generality of them treated him as a redoubtable Magician: and the credulous writers of those times relate many ridiculous stories about him; as that he understood the language of birds; that he could raise the Devil, was very familiar with him, and bequeathed his soul to him after death, &c. &c. See *Vincent's Lectures against Popery*, Lect. VII. p. 191. a book in which many stories of the kind are to be found.

literary hero possessed an enlargement of mind, and a thirst for knowledge, rarely to be met with. He was educated in the monastery of Fleury; but soon discovering the incapacity of his teachers, he fled from his monastery, and went to Spain, which was then under the dominion of the Arabs. Having fixed himself at Corduba, he applied with ardour to the acquisition of the Arabian language, and the sciences which that people almost exclusively possessed; he succeeded so well, that in a few years he returned to France, and enriched the Christian world with the literary spoils obtained from the Mahometans, A. D. 960. To him the nations of Europe are indebted for the most valuable of all his acquisitions, a knowledge of the Arabian numeral figures, on the use of which depends every subsequent improvement in Arithmetic.

The Arabian method of notation introduced from Spain by Gerbert, notwithstanding its advantages, was not so eagerly adopted as one might be led to expect; 150 years having elapsed before it was known in Britain, and nearly 100 more before it was brought into common use, as is shewn by Dr. Wallis.

The first writer of note after the reception of the Arabian method, was Jordanus of Namur in Flanders, about the year 1200; his work was commented on, and published shortly after the invention of printing, by Johannes Faber Stapulensis, viz. in 1480. Johannes de Sacro Bosco, an Englishman, wrote a treatise on Arithmetic in the thirteenth century; as did Maximus

Gerbert was preceptor to Robert I. King of France, and to Otho III. Emperor of Germany. He was Bishop of Rheims, and afterwards Archbishop of Ravenna. At length, on the death of Pope Gregory V. A. D. 998, Gerbert was, by the influence of his pupil Otho, chosen to succeed him on the Papal throne, under the name of Sylvester II. He died about the year 1003.

Planudes, the Scholiast, either in that century or the next.

After the introduction of printing, the diffusion of knowledge necessarily became much more extensive than it had been at any former period, from the number of books which were successively published. The earliest authors who wrote on Arithmetic were Lucas De Burgo, 1470. Cardan, Purbach, Stifelius, Scheúbelius, Tartalea, Maurolycus, Peletarius, &c. these were foreigners. Of our own countrymen, Recorde, Bulkley, Digges, and Dee, were among the earliest writers. The doctrine of Decimal Fractions was introduced about 1464, by Regiomontanus[m]: but the first

[m] John Muller was born at Mons Regius, in Koningsberg, in 1436, and received the name of Regiomontanus from his birth-place, where, and at Leipsic, he acquired the rudiments of Mathematics and Astronomy. At fifteen he went to Vienna, where he studied to good purpose, under the celebrated Purbach, to whom he became a useful assistant, and an affectionate friend. He afterwards accompanied Cardinal Bessarion, the friend and patron of science, to Rome, where our author studied the Greek language, and at the same time continued his Astronomical labours. In 1463 he went to Padua, where he became a member of the University, and explained the works of the Arabian philosopher Alfraganus. Having collected a great number of Manuscripts, he returned to Vienna, and resumed the duties of his office: at length he retired to Noremberg, and set up a press, intending to print and publish the valuable books he had written or collected, and of which the catalogue is still in being. Here he became acquainted with Bernard Walther, a sincere lover of the sciences, who, entering heartily into his views, undertook the expence of erecting a printing-house, and constructing Astronomical instruments. He now printed *The new Theories* of Purbach, *The Astronomicon* of Manlius, *The Cosmography* of Ptolemy, with *select Commentaries on the Almagest;* also *The new Calendar*, and *Ephemerides* of his own composing.

In 1474 Pope Sixtus IV. invited our Author to Rome, to assist in reforming the Calendar. To induce him to leave his retreat, the Pope made him large promises, and nominated him Bishop of Ratisbon. He consented, and arrived at Rome in 1475, but died the next year, as it is supposed, by poison. The atrocious deed is ascribed to the sons of George Trabezond, in revenge for their father's death, who is said to have died of a broken heart, in consequence of some severe criticisms made by Regiomontanus, on his Translation of Ptolemy's Almagest.

who wrote expressly on the subject was Simon Stevinus, of Bruges, about 1582. Dr. Wallis, in 1657, published his mathematical works, wherein he has the first of any treated at large of Recurring Decimals. Some hundreds of books on the subject, possessing various degrees of merit, have from time to time appeared, in many of which the fundamental principles and rules have been laid down with much clearness and perspicuity, and their applications to mathematical, mechanical, and commercial subjects (which were mostly received from the Arabians) simplified, extended, and improved. Omitting a long list of names, we pass on to the next valuable discovery in Arithmetic, namely, the invention of Logarithms, or numbers whereby the most tedious and difficult calculations are performed with surprising ease and facility. For this invention the world is indebted to the skill and industry of John Lord Napier, a Scotch Nobleman, who first published it in 1614; and for a most important improvement in the system, which took place three years after, to Mr. Henry Briggs, Professor of Geometry at Gresham College. Further particulars of this interesting discovery will be given in its proper place; and we will conclude this sketch with the mention of a few names, to one or other of which most of our countrymen are indebted for their skill in the science. The Arithmetic of Mr. Edmund Wingate[a] was first published in 1629; and after, an edition of the same, improved and enlarged by John Kersey[b], teacher of the Mathematics

[a] Mr. Wingate, a zealous cultivator and encourager of mathematical learning, flourished in the reigns of James and Charles the First. He carried the knowledge of Logarithms to France, where he published some Tracts on the subject: he likewise applied the Logarithms to two sliding rulers, so accommodated to each other, that problems may be mechanically performed by them without the assistance of compasses.

[b] Kersey lived in the reign of Charles the Second. He was the author of

in London: this book had a good sale, and was considered as a useful introduction when our grandfathers were boys at school. The arithmetical part of the Young Mathematician's Guide, by Mr. John Ward[p] of Chester, is remarkably plain and clear for the time in which it was written. This work, which appeared in 1706, has been much esteemed, and still maintains its reputation. Mr. Malcolm's New System of Arithmetic, theoretical and practical, published in 1730, is a very complete work, and served as a model to some of our best elementary writers. Dilworth's[q] Schoolmaster's Assistant, 1743, was much in use thirty or forty years ago; it contains an ample collection of easy examples under every rule, and is on the whole a good old-fashioned School-book. Fenning's Arithmetic is a plain and easy system of rules, with very few examples. Walkingame's Tutor's Assistant has had a great run; indeed it has been found more useful to the practical scholar than books more scientifically written. Its proprietors have taken great pains to render the work as perfect as possible: a few alterations in its structure would make it the best school book on practical arithmetic in print. Dr. Hutton's Treatise on Practical Arithmetic needs no better recommendation than his name. The same may be said of Mr. Bonnycastle's Scholar's Guide; in this work the rules are not only exemplified, but demonstrated, and the taste and science of the author appear

an excellent treatise on Algebra in folio, wherein the Diophantine Problems are very skilfully managed; he also wrote an English Dictionary.

[p] John Ward was born in the year 1648. He appears from his manner and style of writing to have been a very respectable scholar, but I know no particulars of his life.

[q] Thomas Dilworth was originally, as I have been informed, an assistant to the Rev. Thomas Dyche, who kept a school at Stratford le Bow: he afterwards was master of a school in Wapping, and published several elementary books, which are still considered as useful.

to great advantage. The questions composed by the late Martin Clare, F. R. S. have been arranged under their proper rules by Mr. Vyse, in a work entitled, The Tutor's Guide, to which he has added a Key, containing the solutions, the whole forming a very comprehensive system. The ingenious Mr. Keith's Complete Practical Arithmetician is very properly entitled; the work together with the Key certainly form the *completest* practical treatise extant: the demonstrations added at the end are very clear and satisfactory, and shew that the author has chosen a very modest title for his work. The Rev. Mr. Joyce's System of Practical Arithmetic, published in 1808, is the last work on the subject which we shall notice; this is a very complete and well-written little book, containing a large collection of well-chosen examples, and much information not to be met with in any other work of this nature.

Arithmetic may be considered as a Science, or an Art: as a Science, it treats of the properties of numbers, of their sums, differences, ratios, proportions, progressions, powers, roots, &c. in the most general and abstracted manner; it considers them purely as numbers, and has no reference to any application or use, except that of deducing one property from another, and constituting a necessary link in the chain of universal science. Although this abstracted consideration of numbers is proper for the mathematician, it will be of little use to the learner; he will find, that the quickest and surest way to gain a good and useful knowledge of numbers is to acquire theory from practice, and apply his theory from time to time as he acquires it to practical purposes.

Arithmetic is to be considered as an Art, when it teaches how to perform operations with numbers, and

to apply them to use in trade and business, and in the common affairs of life. Surely arguments cannot be necessary to prove that no art is more generally useful than this. Whatever our occupations or engagements in life may be, in every trade, business, and employment, to every individual, rich and poor, the knowledge of numbers is necessary. But we need not enlarge on this subject; a small degree of experience and observation will be sufficient to convince the candid enquirer of the great usefulness of Practical Arithmetic.

In commencing his mathematical studies, the learner will begin with [r] Notation; this and Numeration he must endeavour to understand *well*, as what are usually called the four fundamental rules depend immediately on the structure of our excellent system of numbers. Addition and Subtraction follow in order; and next the Multiplication-table, which must be learned sufficiently perfect, that it may be repeated through from one end to the other, either backwards or forwards, without mistake or hesitation. Having acquired a perfect knowledge of the table, Multiplication and Division, which follow next in order, will not be found difficult. To pass through these rules in a blundering and aukward manner, although it may satisfy a lazy dunce, will not be sufficient for him who aspires to knowledge: if any operation is not *perfectly* understood, so as to be performed with tolerable ease, the previous examples ought to be worked over again, and repeated until it is. Having passed through the rules in the order they stand in this book, and occasionally consulted the notes, so as to understand the reasons on which the rules are founded, their connection with each

[r] The word *Notation* is derived from the Latin *nota*, a mark, and *Numeration* from *numerus*, a number.

other, and dependence on self-evident principles, the learner may proceed to Algebra; he will find very little difficulty in that if he understands the arithmetical part well.

DEFINITIONS.

1. AN unit is that which is known by the name of *one*.

2. Number is either an unit; a collection of two or more units; or one or more parts of an unit.

3. A whole number is that which consists of one or more units.

4. A broken number or *fraction* is that which consists of one or more parts of an unit.

5. An even number is that which can be divided into two equal whole numbers.

6. An odd (or uneven) number is that which cannot be divided into two equal whole numbers.

7. An integer is any whole quantity or thing, considered as a *whole*; the word is used in opposition to a *part*.

WHOLE NUMBERS.

8. Arithmetic of whole numbers teaches how to calculate or compute by whole numbers.

9. The fundamental rules of Arithmetic are Notation and Numeration, whence are derived Addition, Subtraction, Multiplication, and Division: in the proper application of these rules the whole art of Arithmetic consists.

NOTATION AND NUMERATION.

10. Notation teaches how to write or express numbers by appropriate characters, either singly, or by a proper combination of two or more characters; and Numeration shews how to read numbers when written.

11. There are ten characters called digits[a] or figures, by one or more of which every number is expressed: they are written

[a] From the Latin *digitus*, a finger. The want of figures to express numbers probably gave rise to digital or manual Arithmetic, in which numbers were expressed, and calculations performed, by the different positions of the hands and fingers. This appears to us a childish play, but it was formerly a serious study, as appears from the elaborate account of it, given by venerable Bede, in his *Opera*, p. 227, &c. Some of the eastern nations still employ this method, and they are said to surpass us in the expedition and accuracy of their calculations.

and named as follows; 1, *one*, or unity; 2, *two*; 3, *three*; 4, *four*; 5, *five*; 6, *six*; 7, *seven*; 8, *eight*; 9, *nine*; and 0, *nought*, (or nothing.)

12. Unity, or *one*, is the least of all whole numbers, and may be considered as the root or origin of all the rest; for if unity be increased by itself, and if the result be increased by unity, and again, if the last result be increased by unity, and so on continually, the several results will constitute the entire system of whole numbers. For example, unity or 1 increased by itself becomes (1, 1, or) 2; again, 2 increased by unity becomes (1, 1, 1, or) 3; in like manner 3 increased by unity becomes (1, 1, 1, 1, or) 4, and so on indefinitely.

13. The nine first numbers are all that can be expressed by single figures; to denote all higher numbers it is necessary to combine two, three, or more figures, and sometimes to employ one or more ciphers.

14. It has been shewn in the preceding article, that all numbers originate in *unity*, and successively arise, by the continual increase of the preceding number by unity, and that the nine figures represent the nine first numbers; also that higher numbers require a combination of two or more figures. Before we explain the method of combination, it will be necessary to shew the manner of classing numbers, which has been universally adopted for the convenience of computation, and is indispensable where high numbers are concerned.

15. Numbers are classed and ranged under the following denominations, viz. *Units*, *Tens*, *Hundreds*, *Thousands*, *Tens of Thousands*, *Hundreds of Thousands*, *Millions*, &c. The first nine numbers constitute the class of *units*: the number which next follows the last of this class (or 9) is ten; this is the first number of the class of *Tens*; this class proceeds thus, (1 ten, or) *Ten*; (2 tens, or) *Twenty*; (3 tens, or) *Thirty*; (4 tens, or) *Forty*; (5 tens, or) *Fifty*; (6 tens, or) *Sixty*; (7 tens, or) *Seventy*; (8 tens, or) *Eighty*; (9 tens, or) *Ninety*; and the next number in this order is (10 tens, or) 1 *Hundred*, which is the first number of the next superior class; this class proceeds thus, 1 *Hundred*, 2 *Hundreds*, 3 *Hundreds*, and so on up to 10 *Hundreds*, which is 1 *Thousand*, or the first number of the next superior class; which in like manner proceeds thus, 1 *Thousand*, 2 *Thousands*, 3 *Thousands*, &c. up to 10 *Thousands*, which is the first number of the next

class superior to the former; this again proceeds, (1 ten Thousands, or) *Ten Thousands;* (2 tens, or) *Twenty Thousands;* (3 tens, or) *Thirty Thousands;* and so on up to 10 ten Thousands, or *One Hundred Thousands*; which is the first of the next superior class; whence proceeding as before we have, 1 *Hundred Thousands*, 2 *Hundred Thousands*, 3 *Hundred Thousands*, &c. up to *Ten Hundred Thousands*, or 1 *Million*, &c. &c.

16. Hence it appears, that 1 ten is *ten units*; 2 tens, *twenty units*; 3 tens, *thirty units*; 4 tens, *forty units*, &c.: that 1 hundred is *ten tens;* 2 hundreds, *twenty tens;* 3 hundreds, *thirty tens*, &c.: that 1 thousand is *ten hundreds;* 2 thousands, *twenty hundreds;* 3 thousands, *thirty hundreds;* and in general that every superior denomination is tenfold the next inferior one; and also that any part of a superior denomination is in like manner tenfold the same part of the next inferior one.

17. It follows, from Art. 12. that there are many intermediate numbers, which, according to the preceding arrangement, must fall under two or more of the foregoing denominations: thus, twenty-five consists of 2 tens and 5 units; six hundred and seventy-eight consists of 6 hundreds, 7 tens, and 8 units; three thousand four hundred and fifty-six consists of 3 thousands, 4 hundreds, 5 tens, and 6 units, &c, &c. Hence a distinct idea of the value of any numbers may be formed from this convenient and beautiful mode of arrangement.

18. Having given a sketch of the general outline, the next thing to be explained is the method of expressing all numbers by the ten digits or figures; in order to which we observe, that each figure, unconnected with any of the other figures, stands merely for its own simple value; but each has besides a local value, namely, a value which depends on the place it occupies when connected with others: thus a figure standing in the first or right hand place expresses only its simple value; but if another figure or the cipher be placed to the right of it, then the figure first mentioned expresses ten times its simple value, that is, as many tens as it contains units. If two figures or ciphers be placed to the right of a figure, that figure expresses ten times what it did when it had only one on its right, or one hundred times its simple value; and so on continually.

19. Hence appears the use of the cipher, which although it is of no value in itself, yet when placed on the right of any num-

ber, it increases the value of that number tenfold; thus 5 standing by itself expresses simply *five;* but if a cipher be placed on its right, thus 50, it then becomes *fifty*, or ten times 5; if two ciphers be placed, thus 500, it becomes *five hundred*, or ten times fifty its former value; let another cipher be placed to the right of the last, and the number becomes 5000, or *five thousand*, which is ten times five hundred, &c.

20. From the two preceding articles, the method of expressing any number by figures may be easily inferred: thus, if it be required to express by figures the number twenty-five, or two tens and five units, it is evident (art. 18.) that five units must be expressed by a 5 in the right hand place of the number to be written, and that the two tens must be expressed by writing a 2 in the second place, or to the left of the 5; thus 25. Six hundred and seventy-eight (or six hundreds, seven tens, and eight units) is expressed by writing 8 in the (right hand or) first place, 7 in the second, and 6 in the third; thus 678: in like manner three thousand four hundred and fifty-six, expressed in figures, is 3456, where the 6 represents 6 units, the 5 five tens, or fifty, the 4 four hundreds, and the 3 three thousands.

21. Numeration, or the reading of numbers, is effected in the following manner; point to the first (or right hand) figure of any number, and call it *units;* point to the second, and call it *tens;* to the third, and call it *hundreds;* to the fourth, and call it *thousands;* to the fifth, and call it *tens of thousands;* to the sixth, and call it *hundreds of thousands;* to the seventh, and call it *millions;* to the eighth, and call it *tens of millions;* to the ninth, and call it *hundreds of millions;* to the tenth, and call it *thousands of millions;* to the eleventh, and call it *tens of thousands of millions;* to the twelfth, and call it *hundreds of thousands of millions*, &c. &c. Then (beginning at the left) read the figures back again from left to right, adding to the name of each figure the denomination you gave it when reading from right to left: in this manner the numbers in the following table are to be read.

NUMERATION TABLE.

Hundreds of Thousands of Millions	Tens of Thousands of Millions	Thousands of Millions	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds	Tens	Units
											9
										9	8
									9	8	7
								9	8	7	6
							9	8	7	6	5
						9	8	7	6	5	4
					9	8	7	6	5	4	3
				9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2	1	0
	9	8	7	6	5	4	3	2	1	0	1
1	2	3	4	5	6	7	8	9	5	6	7

Here the denominations are placed over the figures, those in the first column being *units*, those in the second *tens*, those in the third *hundreds*, &c. wherefore the first line of the table will be *nine* (units), the second *ninety-eight*, the third *nine hundred and eighty-seven*, the fourth *nine thousand eight hundred and seventy-six*, &c. and the last *one hundred and twenty-three thousands four hundred and fifty-six millions, seven hundred and eighty-nine thousands, five hundred and sixty-seven*. When a number contains one or more ciphers, the denominations which the ciphers occupy are to be omitted in reading; thus, 405 is read *four hundred and five*; here are no *tens*: 30 is read *thirty*; here are no *units*: 70003 is read *seventy thousands and three*; here the denominations of *tens*, *hundreds*, and *thousands* are wanting.

22. The method of classing numbers as above explained may be extended to any length: but the most convenient method of assisting the mind to form an idea of large numbers is to divide them into periods of six figures each, beginning at the right, calling the first period *units*, the second *millions*, the third *billions*, &c. according to the following table: where it must be remarked,

that each period contains units, tens, hundreds, thousands, tens of thousands, and hundreds of thousands, of the denomination marked over that period.

QUADRILLIONS.						TRILLIONS.						BILLIONS.						MILLIONS.						UNITS.					
Hundreds of Thousands of	Tens of Thousands of ..	Thousands of	Hundreds of	Tens of	Units of	Hundreds of Thousands of	Tens of Thousands of ..	Thousands of	Hundreds of	Tens of	Units of	Hundreds of Thousands of	Tens of Thousands of ..	Thousands of	Hundreds of	Tens of	Units of	Hundreds of Thousands of	Tens of Thousands of ..	Thousands of	Hundreds of	Tens of	Units of	Hundreds of Thousands of	Tens of Thousands of ..	Thousands of	Hundreds of	Tens of	Units of ....
6	1	7,	8	3	4.	1	3	0,	9	2	7.	6	3	1,	8	2	9.	4	0	3,	1	7	2.	8	9	5,	2	6	3.

The right hand place of each denomination is units of that denomination: but we do not pronounce the word *units* in reading, except at the right hand place of all; instead of it we say, *millions, billions,* &c. naming the right hand figure of each period simply by the denomination marked over that period.

23. When any number expressed in words is required to be expressed in figures, if the learner is at a loss how to do it, he may make as many dots (placing them in a line from right to left) as there are places in the number to be written, calling the right hand dot *the place of units*, the second *the place of tens*, and so on; then under the said place of units put the units figure of the number to be written; under the place of tens put the tens figure of the number; under the place of hundreds put the hundreds figure of the number, &c. and if at last there be any dot without a figure under it, the place must be supplied by a cipher. Thus to write the number four thousand three hundred and fifty-six in figures—here are *units, tens, hundreds,* and *thousands;* four dots · · · · must therefore be made; the left hand dot representing the place of thousands, 4 must be placed under it; under the next dot, or place of hundreds, 3 must be placed; under the next, which represents the place of tens, 5 must be placed; and 6 under the right hand dot, which represents the place of units; thus $\dot{4}\dot{3}\dot{5}\dot{6}$. To write in figures one million two thousand and thirty; here we want the place of

units, tens, &c. up to millions; nine dots will therefore be necessary, thus, · · · · · · · · ·; put 1 in the millions place, 2 in the thousands place, and 3 in the tens place, and you will have 1 2 3 ; then, supplying the vacant places by ciphers, the number will become 100002030, which is what was required.

EXAMPLES.

Write in figures the following numbers.

1. Twenty-four. 2. Three hundred and sixty-two. 3. Seven thousand two hundred and forty. 4. Ninety thousand. 5. Eight hundred and ten. 6. One million and nine. 7. Sixty-seven thousand two hundred and one. 8. Two hundred million three hundred thousand four hundred. 9. One million and sixty-four. 10. Thirty thousand three hundred and thirty-three. 11. Five hundred billions.

Write or express in words the following numbers.

14 . . . 23 . . . 70 . . . 123 . . . 590 . . . 509 . . . 4321 . . . 5040 . . . 1002 . . . 23456 . . . 30405 . . . 987654 . . . 100200 . . . 234567 . . . 9080070 . . . 817263̇54 . . . 7018̇20734 . . . 102003000400000.

24. The following characters are employed to mark the connection of numbers, or to denote certain operations.

The mark + (named *plus*, or more) denotes addition. The mark — (named *minus*, or less) denotes subtraction. The mark × (named *into*) denotes multiplication. The mark ÷ (named *by*) denotes division. The mark √ is called a *radical sign;* and a line drawn over two or more numbers, (serving to connect them,) thus $\overline{3 \times 4}$, is called a *vinculum*. The mark = is the sign of equality. The further use of these characters will be explained in the proper places.

ADDITION.

25. Simple Addition[a] teaches to collect two or more whole numbers into one, which is called their sum.

The mark for Addition is + plus, (*more*,) and shews that the number which follows the sign, is to be added to the number standing before it.

26. *To add single figures together.*

[b] Rule I. Begin at the bottom, and find what number will

[a] The word *Addition* is derived from the Latin *addo*, to put to; *sum* from *summa*; and *proof* from *probo*, to prove, or make out.

[b] This rule depends on the method of notation; (Art. 12.) thus, in the first example, if I want to know the sum of 7 and 4, I must evidently resolve each into the units of which it is composed, and then count all the units in both, one by one, to find the amount; and this practice I must follow until my mind acquires from habit, a sufficient dexterity in numbering to do without it. The only method that a person totally ignorant of Addition could employ, would be this; he would write down all the *ones* in each of the numbers to be added, and then count the whole. Thus ex. 1. would stand according to such a method,

3.	2.	5.	4.	7.
111.	11.	11111.	1111.	1111111.

These *ones* being counted, are found to amount to 21, which is the sum of the given numbers. Simple as this explanation may appear, it is plain that the reason of the rule can be shewn on no other principle. By this method this following Table was first calculated.

1	2	3	4	5	6	7	8	9
2	4	5	6	7	8	9	10	11
3	5	6	7	8	9	10	11	12
4	6	7	8	9	10	11	12	13
5	7	8	9	10	11	12	13	14
6	8	9	10	11	12	13	14	15
7	9	10	11	12	13	14	15	16
8	10	11	12	13	14	15	16	17
9	11	12	13	14	15	16	17	18

This is called *The Addition Table*, and its use is to find readily the sum of any two numbers, each not exceeding 9. Thus, look for the two numbers, viz. one in the left hand column, and the other in the top line, and at the point where the column and line (in which the two numbers are) meet, is the sum of the said two numbers. Thus to find the sum of 4 and 5, look for 4 in the left hand column, and 5 in the top line, and at the point where the two rows of figures (one vertical, the other horizontal) meet stands 9, which is the sum of 4 and 5. To find the sum of 7 and 6, look for 7 on the left, and 6 at top, and at the point where the lines containing 7 and 6 meet stands 13, their sum. In like manner 5 and 9 are found to be 14; 8 and 3 are 11; 9 and 9 are 18, &c. &c. Those who

arise by taking the units in the lower figure, and the units in the next figure above it, into one sum.

II. Do the same with this sum and the third figure—with this last sum and the fourth figure, and so on until all the figures have been taken; set down the last sum between two lines below, and it will be the sum required.

Method of proof. Draw a line under the top figure; then add up all the rest of the figures as before, and place the sum under the former sum; add this last sum and the top figure together, and if the sum is the same as the sum first found, the work is right.

EXAMPLES.

1. Add the figures 3, 2, 5, 4, and 7 together.

OPERATION.	
	3
	2
	5
	4
	7
Sum	21
	18
Proof	21

Explanation.

I first place the given figures in a column under one another; then, beginning at the bottom, I say, 7 and 4 are 11, then 11 and 5 are 16, then 16 and 2 are 18, then 18 and 3 are 21, which (because all the figures have been used) is the sum; I therefore place it at the bottom. Next I cut off the upper figure 3, and, beginning at the 7, I add all up as before, except the 3 cut off, and place the sum 18 below the former. Then I add the last sum 18 to the 3 cut off, and the sum is 21, which being the same as the sum first found, shews that the work is right.

2. Add up the following columns of figures.

	2	4	5	2	1	2	9	9	8
	9	7	4	8	2	4	2	9	7
	8	3	3	1	3	5	8	7	9
	5	8	9	3	4	3	3	6	6
	3	9	7	2	5	1	7	5	4
Sum	27	31	28						
	25	27	23						
Proof	27	31	28						

26 B. *To add any whole numbers together.*

RULE I. Place the numbers under one another, so that units may stand under units, tens under tens, hundreds under hundreds, &c.

II. Add up the figures in the units (or right hand) column by the former rule; take out all the tens from the sum, and set

---

cannot readily add small numbers, ought to learn this Table by heart; thus the method of adding small numbers being once familiar, that of adding larger numbers will be gradually acquired by practice.

down (below the figures added) what is over, or, if nothing be over, set down a cipher.

III. Carry as many units (or ones) to the second column as there were tens in the first; add these up with the second column as before; take out the tens from the sum, set down the remainder, carry 1 for every ten to the third column, and proceed in this manner till the left hand column is added, under which its whole sum must be put down [e].

3. Add 312, 498, 387, 968, and 527 together.

OPERATION.

	312
	498
	387
	968
	577
Sum	2742
	2433
Proof	2742

Explanation.

Having placed the numbers, I find that the sum of the units column is 32, or 3 tens and 2 over; I put down 2, and carry 3 to the second column, the sum of which is 34; I therefore put 4 down, and carry 3 to the last column, the sum of which is 27, which I put down. The second line and proof are done as directed in the last rule.

---

[e] Having explained the method of *adding*, it remains to account for the method of *carrying* prescribed in the rule. Thus, in example 3, the sum of the units is 32, or 3 tens and 2 units; I must evidently put down the 2 units; but it is plain that the 3 tens must be added with the tens, namely, with the second column, which consists of tens: again, the sum of the second column (with the 3 carried) is 34, that is 34 tens, or 340; the 4 tens then must evidently be put down under this second column, (which is tens,) and the 3, which are hundreds, must be collected with (or carried to) the hundreds, the sum of which is 27 hundreds, or (which is the same) 2 thousand 7 hundred; this sum, it is plain, must be put down, as there can be no further carrying. Thus the rule teaches not only to collect several numbers into one, but likewise to class and arrange the different denominations in the sum, by continually reducing lower to higher denominations, as often as a sufficient number of the former arises.

The method of proof in this and the foregoing rule will be easily understood; for having cut off the top line, and added up all the rest of the figures, the result will be the sum, exclusive of the top line; wherefore if the said result and top line be added together, the number thence arising will evidently be the same as the sum, or upper line of the work.

Add the following sums.

	4.	5.	6.	7.	8.
	1234	2357	3142	4135	5341
	4123	3512	2314	5413	3415
	5412	5124	4231	2541	4153
	3541	1243	1423	1254	1536
	1354	2431	3142	4125	5361
Sum	15664	14667			
	14430	12310			
Proof	15664	14667			

9. Add the numbers 4321, 8037, 2345, 6728, and 1091 together. *Sum* 22522.

10. Add 109, 1237, 34, 987, and 12 together. *Sum* 2379.

11. Add nine thousand eight hundred and sixty-seven to the sum of the following numbers, 98, 876, 129, 9086, and 12345. *Sum* 32401.

12. A has 39 marbles, B has 68, C 24, D 190, E 59, and F 95; how many have they among them? *Ans.* 475 *marbles.*

13. Received of G 12 shillings, of H 45, of K 130, of L 679, of M 99, and of N nine hundred and ninety-nine; how many did I receive in all? *Ans.* 1964 *shillings.*

14. A person to maintain himself and family 5 years spent as follows; viz. the first year 687l. the second, 989l. the third, 836l. the fourth, 1094l. and the fifth, 1209l. what did he spend in all? *Ans.* 4815*l.*

SUBTRACTION.

27. Simple Subtraction[d] teacheth to take a less whole number from a greater, whereby the remainder or difference is known.

The mark for Subtraction is —; it is named *minus*, (or *less*,) and shews that the number following the sign is to be taken from the number which stands before it.

RULE I. Place the less number below the greater, and let units stand under units, tens under tens, &c. as in Addition, and draw three lines at proper intervals below.

II. Begin at the right hand figure in the less number, and take

---

[d] The name *Subtraction* comes from the Latin *sub* under, and *traho* to draw.

it out of the figure above, and set what remains under it: do the same with all the figures in the less number, setting each remainder under the figure from whence it arises.

III. But if it happens that a figure in the lower line be greater than that above it, add ten to the upper one, after which take the lower figure from the sum; set down the remainder, and carry one to the next lower figure before you subtract.

IV. Proceed in this manner until all the lower figures are subtracted, and the result will be the remainder, or difference required.

Method of Proof. Add the difference found and the less number together, then if the sum be equal to the greater number, the work is right [e].

---

[e] To shew that this rule has its foundation in Notation, let it be required to take 3 from 7; now if we represent the 7 by its units, agreeable to art. 12, and cut off from these the number of units in 3, the rest of the units being counted will shew what remains, thus, 1, 1, 1 | 1, 1, 1, 1, where having cut off 3, the rest of the units being counted I find amount to 4; therefore 3 taken from 7, 4 remains; and the same may be shewn of other numbers.

When each figure in the lower line is less than its correspondent figure in the upper, it is plain, that by taking each lower figure from that above it, we obtain the several differences of all the parts, which, taken together in order, will evidently constitute the difference of the whole.

But when any figure in the upper line is less than its correspondent one in the lower, we borrow 10, which is evidently 1 of the next higher denomination, after which we carry 1 (to make up for the borrowing) to the lower figure of the next higher denomination; by thus increasing the number to be subtracted, we eventually take the 1 away from that denomination from whence it had before been nominally borrowed.

For the method of proof. If a less number be taken from a greater, what remains will be the difference; and if the difference of these two numbers be added to the less, the sum will be the greater: this is too plain to require illustration; we will barely apply it to the first example, where the upper line is the greater number, the second line the lesser, the third line the difference, and the fourth line (which is equal to the greater) is the sum of the difference and less number.

The Table in the note on art. 26. may be made a *Subtraction Table:* thus to find the difference of two numbers, look for the least at top, and in the same column find the greatest; then the number in the left hand column, which stands in the same line with the greatest, is the difference required: thus to take 7 from 12, look for 7 at top, and 12 in the same column, then opposite 12 in the left hand column stands 5, the difference.

EXAMPLES.

1. From 4738 take 2123.

OPERATION.	
	4738
	2123
Diff.	2615
Proof	4738

Explanation.

Having placed the less number under the greater, I take 3 from 8, and 5 remain to put down; then 2 from 3, and 1 remains; 1 from 7, and 6 remain; 2 from 4, and 2 remain; the whole remainder then is 2615. I then add the second line 2123 and this remainder together, which gives the proof.

2. From 135167 take 19103, and from 720981 take 10029.

OPERATIONS.		
	135167	720981
	19103	10029
Diff.	116064	710952
Proof.	135167	720981

Explanation.

In the first of these operations the work is easy, till we come to the 9, where we say 9 from 5 I cannot, but borrowing 10, and adding it to the 5, the sum is 15; therefore 9 from 15, and 6 remain; put down 6, and carry 1 to the next figure 1, which makes 2, then 2 from 3, and 1 remains to put down. There being no figure under the left hand figure 1, I say 0 from 1, and 1 remains to put down. In the second operation, I say 9 from 1 I cannot, borrow 10, then 9 from 11, and 2 remain to put down; then carry 1 to the 2 makes 3, 3 from 8, and 5 remain to put down: the rest as before.

	3.	4.	5.	6.
From	37485	30748	48005	607346
Take	12301	10091	2301	59840
Diff.				
Proof				

7. If from one thousand two hundred and thirty-four, seven hundred and eighty-nine be taken, what will be left? *Ans.* 445.

8. If three hundred and sixty-five be taken from five hundred and sixty-three, what is the remainder? *Ans.* 198.

9. If one thousand two hundred and thirty-four wheat corns be taken out of a bin containing one million, how many will be left? *Ans.* 998766.

10. A man was born in the year 1767, and died in 1799; what was his age? *Ans.* 32 *years.*

11. One was born in 1773, and another in 1801; required the difference of their ages? *Ans.* 28 *years.*

12. The art of printing was discovered in 1449, how many years is it since, this being 1812? *Ans.* 363 *years.*

13. A courier travelled three thousand miles in one year, and only one thousand nine hundred and nine in the next; how much does the former distance exceed the latter? *Ans.* 1091 *miles.*

14. Out of a thousand pounds, a person paid away eight hun-

dred and fifty-two; how many pounds had he remaining? *Ans.* 148 *pounds.*

15. Borrowed one thousand two hundred and thirty-four guineas, and paid in part nine hundred and eighty-seven; what sum is there still remaining due? *Ans.* 247 *guineas.*

16. There is a person, who if he lives until the year 1820 will be seventy-five years old; in what year was he born? *Ans. in* 1745.

17. King George the Second came to the throne in 1727, and died in 1760; how long did he reign? *Ans.* 33 *years.*

18. The western empire was destroyed by Odoacer, king of the Heruli, in the year 476, and the eastern empire submitted to Mahomet the Second, emperor of the Turks in 1453; how many years did the latter exist after the former? *Ans.* 977 *years.*

MULTIPLICATION.

28 Multiplication[f] is a short method of addition; it teaches how to find the sum that arises from repeating one number, called the *multiplicand,* as often as there are units in another number, called the *multiplier.*

The number sought, or that which arises from the operation, is called the *product.*

The multiplicand and multiplier are frequently called *Terms* or *Factors,* and the product is sometimes called the *Factum.*

The mark denoting Multiplication is $\times$; it is named *into,* and shews that the number standing before the sign is to be multiplied *into* (or by) the number which follows it; thus 3×4 denotes that 3 is to be multiplied by 4, or taken 4 times[g].

To perform the operations in this rule with ease, a table of the products of every two numbers, each not exceeding 12, has been contrived; it is called the Multiplication Table, and must be well understood, learned by heart, and remembered.

---

[f] The name *Multiplication* comes from the Latin *multus* many, and *plico* to fold; *product* from *produco* to produce; *factor* a maker or doer, and *factum* a thing done or made.

[g] When two or more numbers are to be multiplied by any number, the *vinculum* is placed over all the former, and a point is frequently interposed between the factors instead of the sign $\times$; thus $\overline{3+4}.5$ denotes that the sum of 3 and 4 (or 7) is to be multiplied by 5; also $\overline{3+7+2}.\overline{6-4}$ shews that the sum of 3, 7, and 2, (viz. 12,) is to be multiplied by the difference of 6 and 4, (viz. 2;) and the like in other cases.

THE MULTIPLICATION TABLE[h].

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Explanation of the Table.

To find the product of two numbers, look for one of them in the top line of the Table, and for the other in the left hand column; then the number which stands directly under the first, and level with the second, is the product required: thus to find 3 times 5, look for 3 at top, and 5 on the left, then under 3 and level with 5 stands 15, which is the product of 3 and 5. To find 10 times 9, under 10 and level with 9 stands 90, the product. To find 6 times 11, under 6 and level with 11 stands 66, the product, &c.

---

[h] To shew that Multiplication is derived immediately from Notation, let it be required to find 3 times 4; put down the units in 4—three times, and then count the whole, (Art. 12.) thus, 1, 1, 1, 1 .... 1, 1, 1, 1 .... 1, 1, 1, 1; these counted amount to 12, therefore 3 times 4 are 12; and the like may be shewn in all cases.

But the rule is commonly derived from Addition thus; to find the product of 3 times 4, put 4 down three times, and find the sum. So if I want to find 9 times 8, I must put nine eights under each other, add them together, and the sum will be 72, or the product of 9 times 8; by this method the Multiplication Table was first formed.

$$\begin{array}{r} 4 \\ 4 \\ \underline{4} \\ 12 \end{array}$$

Simple Multiplication, or Multiplication of whole numbers, is performed by the following rules.

29. *When the multiplier does not exceed* 12.

RULE I. Under the right hand figure of the multiplicand write the multiplier.

II. Multiply every figure in the multiplicand by the multiplier, and set the product, if it be less than 10, under the figure multiplied.

III. But if the product be 10, or more, set down the units only, and carry 1 for every 10 to the next; multiply the next figure, and *after you have multiplied it*, (not before,) carry the *ones* to the product.

IV. Proceed in this manner till all the figures are multiplied; at the last (or left hand) figure, the whole product must be set down[i].

---

[i] The sum of numbers is evidently of the same denomination with the numbers added; it cannot be of any other; consequently the product of any number multiplied by another must be of the same denomination with the number multiplied. Hence if units be multiplied by any whole number, the product is units; if tens be multiplied, the product will be tens; if hundreds be multiplied, the product will be hundreds, &c. Let it be required to multiply 574 by 4; now the 4 units multiplied by 4 produces 16 units; the 7 tens multiplied by 4 produces 28 tens, or 280; and the 5 hundreds multiplied by 4 produces 20 hundreds, or 2000. Wherefore these several products when added together will give the product of the two given numbers.

Thus 574	*The same by Addition*	*By the Rule*
4	574 ⎫ where the	574
16 = 4 × 4	574 ⎬ given num-	4
280 = 70 × 4	574 ⎪ ber is taken	2296
2000 = 500 × 4	574 ⎭ 4 times.	
2296 = 574 × 4	2296	

From this example, the truth and reason of the Rule will be manifest. With respect to the method of carrying, it is plain from Notation that 10 units are 1 ten, 20 units 2 tens, &c. 10 tens are 1 hundred, 20 tens 2 hundreds, &c. And in general, any number of *tens* of an inferior denomination will be so many *units* of the next superior; and therefore 1 is always carried for every ten, from the inferior to the next superior denomination, as directed in the Rule.

EXAMPLES.

1. Multiply 375294 by 2.

OPERATION.

Multiplicand	375294
Multiplier	2
Product	750588

Explanation.

Having written the multiplier 2 under the right hand figure of the multiplicand, namely, under the 4, I begin by multiplying that figure; thus I say twice 4 are 8, and put it down; next I say twice 9 are 18, put down 8 and carry 1; then twice 2 are 4 and 1 I carried 5, put down 5; then twice 5 are 10, put down 0 and carry 1; then twice 7 are 14 and 1 carried 15, put down 5 and carry 1; lastly, twice 3 are 6 and 1 carried 7, I put it down, and the work is finished.

2. Multiply 968754 by 3.

OPERATION.

Multiplicand	968754
Multiplier	3
Product	2906262

Explanation.

Here I say 3 times 4 are 12, put down 2 and carry 1; 3 times 5 are 15 and 1 carried make 16, put down 6 and carry 1; 3 times 7 are 21 and 1 carried 22, put down 2 and carry 2; 3 times 8 are 24 and 2 carried 26, put down 6 and carry 2; 3 times 6 are 18 and 2 carried 20, put down 0 and carry 2; lastly, 3 times 9 are 27 and 2 carried make 29, I put down the whole 29, and the work is finished.

	3.	4.	5.	6.
Mult.	2793812	7849265	5381497	8376491
By	4	5	6	7
Prod.	11175248	39246325	32288982	58635437

	7.	8.	9.	10.
Mult.	2738469	3182735	4170826	5742873
By	2	3	4	5
Prod.				

	11.	12.	13.	14.
Mult.	6179084	7410826	8918292	9123847
By	6	7	8	9
Prod.				

	15.	16.	17.	18.
Mult.	1982745	2936807	3869071	4965723
By	2	3	4	5
Prod.				

	19.	20.	21.	22.
Mult.	5843672	6172083	7123964	8076945
By	6	7	8	9
Prod.				

	23.	24.	25.	26.
Mult.	9138765	1920867	2374563	3948657
By	11	11	12	12
Prod.				

	43.	44.
Multiply	273814	381725
By	4567	6789
	1916698	3435525
	1642884	3053800
	1369070	2672075
	1095256	2290350
Product	1250508538	2591531025

32. *To prove the truth of these operations by casting out the nines.*

RULE I. Having made a cross like the sign for multiplication, add the figures in the multiplicand together, rejecting every nine as it arises, and reserving only the remainder to carry: set the last remainder on the left of the cross.

II. Add the figures in the multiplier together, casting out the nines as before, and set the last remainder on the right of the cross.

III. Multiply the two remainders together, and, having cast out the nines from the product, set the remainder at the top of the cross; if there are no nines, the product itself must be set.

IV. Lastly, cast the nines out of the product, and set the remainder at the bottom of the cross: if the bottom and top figures are alike, the operation is right, except you have made a mistake of nine exactly; but if they are not alike, the work is certainly wrong.

Thus to prove Ex. 43. I say, 2 and 7 are 9; this I omit; and go on, 3 and 8 are 11; (omitting the 9) I carry 2 to the 1, which make 3, and 4 are 7; this I put down on the left of the cross.

$$\begin{array}{ccc} & 1 & \\ 7 & \times & 4 \\ & 1 & \end{array}$$

Next, beginning with the multiplier, I say, 4 and 5 are 9, which I omit; and go on, 6 and 7 are 13, which is 4 above 9; omitting the 9, I put down 4 on the right side of the cross.

Thirdly I multiply 7 by 4, and the product is 28; in this number I find there are 3 nines (which I omit) and 1 over; therefore I put 1 at the top.

Lastly, I cast the nines out of the product; thus, 1 and 2 are 3 and 5 are 8 and 5 are 13, which is 9 and 4 over; (omitting the 9) I carry 4 to the 8 are 12, which is 9 and 3 over; (omitting the 9) I carry 3 to 5 are 8 and 3 are 11, which is 2 above 9 (omitted); carry 2 to 8 are 10, which is 1 above 9; this 1 I set down at the bottom of the cross, and finding that it agrees with the figure at the top, I conclude that the operation is right, except I have made a mistake of 9.

The proof of Ex. 44 is $\begin{array}{ccc} & 6 & \\ 8 & \times & 3 \\ & 6 & \end{array}$ which is done exactly in the same manner.

33. *When ciphers are intermixed with the other figures in the multiplier*, they need not be regarded, provided the first figure of each line in the multiplying, be put under the figure you multiply by[m].

```
            45.      Proof.                 46.      Proof.
Multiply 830724     \ 3 /                 312905    \ 4 /
      By  23021    6  X  8                210035   2  X  2
         830724     / 3 \                1564525    / 4 \
        1661448                          938715
       2492172                          312905
      1661448                          625810
Prod. 19124097204                Prod. 65721001675
```

Explanation.

Ex. 45. Although here are 5 figures in the multiplier, yet one being a *cipher*, there are only 4 figures to multiply by: the difference made by the cipher is, that it occasions the third and fourth lines to stand one place more to the left, than they would if the cipher was not there.

Ex. 46. Here are 6 figures in the multiplier, but only 4 multiplying figures; the two ciphers remove the third and fourth lines two places to the left.

In examples of this kind, when ciphers are subjoined to the right of both factors, they are to be omitted in the multiplying, and all of them are to be written to the right hand of the adding.

```
            47.     Proof.                  48.      Proof.
Multiply 1329470    \ 2 /                 210300    \ 0 /
      By     340   8  X  7                 57000   6  X  3
          531788    / 2 \               14721       / 0 \
         398841                        10515
Prod.  452019800                 Prod. 11987100000
```

Explanation.

In Ex. 47, I multiply by 34, and take no notice of the two ciphers in the multiplying part; but I bring them both down to the right hand of the adding.

Ex. 48. Here are 5 ciphers subjoined to both factors; they are omitted in the work, and set down at the end of the adding, as before.

34. When the figures 11 or 12 stand together in a multiplier, the multiplication by both figures may be performed in one line, (see Ex. 24. 25. Art. 29.) provided the right hand figure of every product stand under the units place of its respective multiplier, as before directed[n].

---

[m] This rule is sufficiently evident from the note on Art. 31.

[n] This rule depends on the reasons given in the note on Art. 31.

	49.	Proof.
Multiply	370954	0
By	11412	1 0
	4451448	0
	1483816	
	4080494	
Prod.	40957773048	

	50.	Proof.
	128643	3
	121211	6 8
	1415073	3
	1543716	
	1543716	
Prod.	15592946673	

Explanation.

Ex. 49. In this example the multiplier consists of 11, 4, and 12. I multiply first by 12, putting the right hand figure 8 of the product under the units 2. I next multiply by 4, putting the right hand figure 6 of the product under the said multiplier 4. Lastly, I multiply by 11, putting the right hand figure 4 of the product under the right hand 1 of the 11. And note, that always in multiplying by a double figure, the units place of the product must stand under the right hand figure of such multiplier.

Ex. 50. Here the multipliers are 12, 12, and 11. The truth of these operations may be proved by multiplying by every figure singly (Art. 31.), or by changing the places of the factors, viz. multiplying the multiplier by the multiplicand.

35. *When the multiplier is a composite number*, that is, the product of two or more numbers in the table.*

Rule. Multiply by one of the component parts, and multiply the product by another, and this last product by another, and so on, when there are several component parts; but this rule is seldom applied when the multiplier is found to consist of more than two parts, or three at most[p].

The operations may be proved by Art. 31.

---

* A number which is the product of two or more numbers (each greater than unity) is called a *composite number*; and the numbers of which it is the product are called its *component parts*. Thus 45 is a composite number, the component parts of which are 9 and 5; for $9 \times 5 = 45$. The terms *composite* and *component* are derived from the Latin *con*, with, and *pono*, to place.

[p] Suppose it were required to multiply 234 by 14: by Art. 28, if I add 234 taken successively 7 times, the sum will be the same as 234 multiplied by 7. Now if I double this sum, or multiply it by 2, the result will be the same as though I had taken down 234 14 times, and added the whole together; that is, 234 multiplied by 14 is the same as if it were multiplied by 7 and the product multiplied by 2. Again, if 234 be multiplied by 2, and the product be taken 7 times, the result will evidently be the same as it would had I taken the given number 14 times; and the same thing may be shewn in every similar case. The truth of the rule may likewise be proved by Art. 31.

51. **Multiply 63958 by 24.**

Here we have a multiplier 24, which may be divided into two component parts a variety of ways; thus 3 × 8, or 4 × 6, or 2 × 12; and these may be taken in a different order, thus 8 × 3, 6 × 4, and 12 × 2; wherefore the operation may be performed, according to this rule, six different ways: thus,

First, by 3 × 8.	*Secondly, by* 4 × 6.	*Thirdly, by* 2 × 12.
63958	63958	63958
3	4	2
191874	255832	127916
8	6	12
1534992	1534992	1534992

Fourthly, by 8 × 3.	*Fifthly, by* 6 × 4.	*Sixthly, by* 12 × 2.
63958	63958	63958
8	6	12
511664	383748	767496
3	4	2
1534992	1534992	1534992

By Art. 31.	*Proof.*
63958	6
24	4 6
255832	6
127916	
1534992	

52. Multiply 52794 by 21; viz. 3 × 7 and 7 × 3. *Product* 1108674.

53. Multiply 28537 by 35; viz. 5 × 7 and 7 × 5. *Product* 998795.

54. Multiply 81764 by 48: viz. 6 × 8 ... 8 × 6 ... 4 × 12, and 12 × 4. *Product* 3924672.

55. Multiply 39468 by 132; viz. 11 × 12 and 12 × 11, *Product* 5209776.

56. Multiply 13896 by 1728; viz. 12 × 12 × 12. *Product* 24012288.

36. *When the multiplier is any number between* 12 *and* 20, *the operation may be performed in one line.*

RULE. Multiply by the right hand figure, and (besides what you carry) to each particular product add the figure which stands next on the right (in the multiplicand) of that which you multiplied; if the last product be a single figure, the left hand

figure of the multiplicand must be placed on its left; but if it be a double figure, the left hand figure of the multiplicand must be added to its last, or left hand figure [q]. The operations to be proved by Art. 31.

	57.	*Proof.*
Multiply	351482	2
By	13	5 × 4
Prod.	4569266	2

Explanation.

I say 3 times 2 are 6, and put it down; then 3 times 8 are 24 and 2 (the right hand figure) 26, put down 6 and carry 2; then 3 times 4 are 12 and 2 carried 14 and 8 (the right hand figure) 22; put down 2 and carry 2; 3 times 1 are 3 and 2 carried 5 and 4 (the right hand figure) 9, put it down; 3 times 5 are 15 and 1 (the right hand figure) 16, put down 6 and carry 1; lastly, 3 times 3 are 9 and 1 carried 10 and 5 (the right hand figure) 15; put down 5, and to the 1 carried add the last figure 3; this makes 4, which I put down in the last place.

	58.	*Proof.*	59.	*Proof.*
Multiply	743896	5	524183	3
By	14	1 × 5	15	5 × 6
Prod.	10414544	5	7862745	3

	60.	61.	62.
Multiply	647184	813625	471283
By	16	17	13
Prod.			

	63.	64.	65.
Multiply	314762	248937	127968
By	14	18	19
Prod.			

PROMISCUOUS EXAMPLES FOR PRACTICE.

66. Multiply 371948 by 28 three ways. (Art. 35, 31.) *Product* 10414544.

67. Multiply 185974 by 14 four ways. (Art. 31, 35, 30.) *Product* 2603636.

68. Multiply 671612 by 114 two ways. (Art. 31, 34.) *Product* 76563768.

69. Multiply 230605 by 819000. *Product* 188865495000.

70. Multiply 128121 by 72001. *Product* 9224840121.

71. Multiply 241643 by 1212 two ways. *Product* 292871[illegible].

---

[q] In this rule the operation of adding continually the back figure amounts exactly to the same as though the top line were taken down one place to the left and added; which is the process directed in Art. 33. by that article it is recommended that each of these examples should be proved.

72. Multiply five million forty-six thousand and one, by four thousand and eight. *Product* 20224372008.

73. Multiply eight hundred and seventy thousand four hundred and ten, by two thousand and fifty. *Product* 1784340500.

74. Multiply nine thousand eight hundred and seven, by nine hundred and eight. *Product* 8904756.

75. Find 42378×100, and 4237800×30000.

76. Find 385746×463, and 178600398×1200.

77. Find 3526×2534, and 8250704×301.

78. Find 10802×10203, and 12345×6789.

79. A beggar collects 478 halfpence in a week; how many halfpence is that a year? *Ans.* 24856.

80. There are 12 signs in the ecliptic, and every sign contains 30 degrees; how many degrees does the ecliptic measure? *Ans.* 360.

81. A clerk calls at 14 places, and receives at each 120l.; how many pounds does he carry home? *Ans.* 1680.

82. The workmen on an estate eat 2167 penny cakes in a day; how many will be sufficient to serve them 128 days? *Answer* 277376.

DIVISION.

37. Simple Division[r] teaches to find how often one whole number, called *the divisor*, is contained in another whole number, called *the dividend*, or to divide a whole number into any proposed number of equal parts; and is a short method of performing continual subtraction[s].

The number arising from the operation is called *the quotient*; it shews how often the divisor is contained in the dividend, that is, into how many equal parts the dividend is divided.

---

[r] The term *Division* comes from the Latin *divido* to divide, distribute, or part; and *quotient* from *quoties*, how many times; also *remainder* from *remaneo*, to tarry behind.

[s] This rule, like all the former, is derivable immediately from Notation. Thus, to divide 12 by 5, resolve the 12 into its constituent units; then count off as many fives from these as you can, putting a stroke between the divisions; there will then be as many fives in 12 as there are divisions; thus, 1.1.1.1.1. | 1.1.1.1.1. | 1.1. here are two complete divisions, and 2 units over, therefore there are 2 fives in 12 and 2 remainder; and the like may be shewn in every instance under this rule.

If any thing be over after the division is performed, it is called *the remainder*.

The mark for division is ÷; it is named *by*, and shews that the number standing before the sign is to be divided by the number which follows it.

37. B. *When the divisor does not exceed* 12.

RULE I. Write down the dividend with a small curve line at each end, draw a line below, and place the divisor on the left hand of the dividend.

II. Find how often the divisor is contained in the first or left hand figure of the dividend, or (if it be not contained in that) in the two or (if necessary) three first figures.

III. Set the quotient, or number denoting *how many times*, below the number divided.

IV. If there be any remainder (it will be less than the divisor,) carry as many tens to the next figure as there are *ones* in the remainder.

V. Divide the sum as before, set down the quotient underneath, carry tens for the units in the remainder to the next figure, divide, and so on until the work is finished, and if there be a remainder at last it may be placed to the right over the divisor, with a small line between them[t].

[t] We here suppose the dividend resolved into parts or denominations; then, beginning at the *superior* denomination, we find by trials how many times the divisor is contained in it; if there be a remainder, we increase the next inferior denomination by it, observing (according to the established principles of Notation) that every unit of the superior becomes ten of the next inferior; in this manner we proceed through all the parts or denominations in the dividend. Thus let 963 be divided by 3; here the dividend resolved into its constituent denominations is 900 + 60 + 3; now there are 300 threes in 900, 20 threes in 60, and 1 three in 3; therefore the quotient will be 300 + 20 + 1, or 321. Let 573 be divided by 4; this number resolved into parts, of which each of the superior denominations is a multiple of 4, will be 400 + 160 + 13, each of which divided by 4, we obtain 100, 40, and 3 for the quotients, with 1 remaining from the last division; therefore 100 + 40 + 3 or 143 is the quotient, and 1 the remainder. We can easily prove that these are respectively the true quotients; for (since multiplication and division are converse rules, it follows that) the quotient multiplied by the divisor, with the remainder added in, will give the dividend; thus, 321 × 3 = 963, and 143 × 4 + 1 = 573, which are the proposed dividends; and since this is the mode of operation prescribed in the rule, it is shewn to be right.

Method of Proof.

Multiply the quotient by the divisor, and to the right hand figure of the product add the remainder; the result will be like the dividend, if the work is right.

EXAMPLES.

1. Divide 351731 by 2.

OPERATION.

2)351731(

Quotient 175865$\frac{1}{2}$

Proof 351731

Explanation.

Having placed the divisor 2 to the left hand of the dividend, I begin thus; twos in 3 will go once, and 1 over; I set the 1 under, and carry the 1 remainder, calling it 10, to the 5, which makes 15; then I say, twos in 15 will go 7 times, and 1 over; put down 7 and carry 1; I call this 1 ... 10, and carry it to the 1, which makes 11, then I say twos in 11 will go 5 times, and 1 over; put down 5 and carry 1, that is 10, to the 7, making 17; twos in 17 will go 8 times, and 1 over; put down 8 and carry 1, or rather 10, to the 3, which make 13; then twos in 13 will go 6 times, and 1 over; put down 6 and carry 1, or 10, to the 1, which make 11, twos in 11 will go 5 times, and 1 over; put the 5 under, and the 1 remainder above the 2 divisor, making $\frac{1}{2}$.

To prove the operation, I multiply the lower line, or quotient, by 2 the divisor, adding 1, the remainder, to the right hand of the product, and the result agreeing with the dividend, (or top line,) shews that the work is right.

2. Divide 198412 by 3.

OPERATION.

3)198412(

Quotient ·66137$\frac{1}{3}$

Proof 198412

Explanation.

I say threes in 1 will not go; I put a dot under, and carry the 1 as 10 to the 9, which make 19, then threes in 19 will go 6 times, and 1 over, and so on. From the last figure but one, viz. the 1, there are 2 over, which in carrying I call 20.

	3.	4.	5.	6.
	4)543235(	5)267130(	6)812415(	7)150125(
Quot.	135808$\frac{3}{4}$	·53426	135402$\frac{3}{6}$	·21446$\frac{1}{7}$
Proof	543235	267130	812415	150123

7.	8.	9.	10.
2)451324(	3)612357(	4)301230(	5)912561(

11.	12.	13.	14.
6)131824(	7)817243(	8)980726(	9)109278(

15.	16.	17.	18.
2)135240(	3)321017(	4)900803(	5)318425(
19.	20.	21.	22.
6)701246(	7)369214(	8)543210(	9)837612(
23.	24.	25.	26.
8)764009(	9)300132(	11)231806(	12)493807(

38. *When the divisor is a composite*[w] *number.*

Rule. Divide by one of the component parts, and the quotient by another, and (if there are more than two) this last quotient by a third, and so on till you have divided successively by all the component parts; the last quotient is the answer[x].

To estimate the remainder when the divisor consists of two component parts. Rule. Multiply the first divisor by the last remainder, and add the first remainder to the product; the result will be the remainder, which may be placed over the original divisor, as directed in the preceding rule, (Art. 37. B.)[y]

---

[w] A composite number is that which is the product of two or more numbers, each greater than unity, which latter are called its component parts.

[x] Division being the converse of Multiplication, the truth of this rule will be evident from Art. 35. To make it still plainer, we observe, that in ex. 27. the dividend, which is required to be divided by 14, is divided by 2, and the quotient by 7. Now dividing by 2 gives the half, and dividing the half by 7 gives the seventh part of the half, which is evidently the fourteenth part of the whole; whence dividing successively by 2 and 7 is equivalent to dividing by 14. And the same may be shewn in all other cases.

[y] The method of proof is thus accounted for from ex. 27. where every unit in the first quotient evidently contains 2 units of the dividend; consequently each unit in the remainder will likewise be equivalent to 2; and therefore the second remainder must be multiplied by 2, (to make it units of the dividend,) making 10, to which adding 1 (unit of the dividend), the remainder is 11.

Likewise in ex. 29, every unit in the first quotient may be considered as containing 11 units of the dividend; every unit in the second, 11 units of the first; and every unit in the third, 11 units of the second; therefore by the foregoing observations $7 \times 11 + 7 = 84$ the remainder, supposing the two last operations

39. *If the divisor consist of several component parts*, the remainder is found as follows; Multiply the last remainder by the preceding divisor; add the preceding remainder to the product, multiply this sum into the next preceding divisor, and add the next preceding remainder, and so on; the last sum is the remainder, and may be placed over the whole divisor as before.

The method of proof is by changing the order of the divisors, that is, by performing the operation again, dividing by that number *first* which you divided by *last* in the preceding operation; if both quotients agree, the work is right.

27. Divide 824135 by 14, viz. by 2×7 and 7×2.

OPERATIONS.

First, 2)824135(1
7)412067(5
Quot. $.58866\frac{11}{14}$
$2\times5+1=11$ *Rem.*

Second, 7)824135(4
2)117733(1
Quot. $.58866\frac{11}{14}$
$7\times1+4=11$ *Rem.*

Explanation.

In the first operation, I divide first by 2, and the quotient by 7. I place each remainder to the right opposite its respective dividend; and then to find the true remainder, I multiply the first divisor 2 by the last remainder 5, (which make 10,) and add the first remainder 1, making 11 for the whole remainder, which is placed over the divisor at the right hand of the quotient.

In the second operation, I divide first by 7, then by 2. To find the remainder, I multiply 7 by 1, and add 4 to the product, which gives 11 for the remainder as before.

Both quotients agreeing, and likewise both remainders, the work is right.

28. Divide 813713 by 72 four ways; viz. 6×12 ... 12×6 ... 8×9, and 9×8.

6)813713(5
12)135618(6
.11301
$6\times6+5=41$ *Rem.*

12)813613(5
9).67809(3
11301
$3\times12+5=41$ *Rem.*

8)813713(1
9)101714(5
.11301
$5\times8+1=41$ *Rem.*

9)813713(5
8).90412(4
11301
$4\times9+5=41$ *Rem.*

---

only. Wherefore also $84\times11+9=933$ the remainder, supposing all three operations: and the same may be shewn for four, or any number of successive operations.

Another method of finding the remainder, is by multiplying the quotient by the divisor, and subtracting the product from the dividend; the result will be the true remainder.

29. Divide 904682 by 1331; viz. by $11 \times 11 \times 11$.

OPERATION.

11)904682(9

11).82243(7

11) .7476(7

Quotient .679$\frac{933}{1331}$

Explanation.

The divisors being all alike, the operation cannot be varied. To get the remainder, I multiply the lower 7 by the middle 11, and add the upper 7 to the product, which gives 84. I next multiply 84 by the upper 11, and add in the 9, which gives 933 for the remainder; thus $7 \times 11 + 7 = 84$. Then $84 \times 11 + 9 = 933$.

30. Divide 316718 by 15, or 3×5 and 5×3. *Quotient* 21114$\frac{8}{15}$.

31. Divide 801234 by 28, or 4×7 and 7×4. *Quotient* 28615$\frac{14}{28}$.

32. Divide 814776 by 66, or 6×11 and 11×6. *Quotient* 12345$\frac{6}{66}$.

33. Divide 979488 by 96, or 12×8 and 8×12. *Quotient* 10203.

40. *When there are ciphers at the right hand of the divisor.*

RULE. Cut off the ciphers by a thin straight stroke drawn between them and the other figures; and cut off as many figures from the right hand of the dividend as there are ciphers cut off from the divisor; then divide the remaining figures in the dividend by the remaining figures in the divisor.

If nothing remain after the operation, the figures cut off from the dividend will be the remainder; but if any thing remain, the figures cut off must be placed to the right of it for a remainder [*].

The method of proof is by multiplying the quotient by the divisor, and adding in the remainder; or by dividing by the component parts of the divisor when it can be done.

[*] Cutting off one figure from both terms is equivalent to dividing both by 10; cutting off two figures from each is equivalent to dividing both by 100, &c. as is obvious from the method of Notation; and it is plain, that as often as the whole divisor is contained in the whole dividend, so often must any part of the divisor be contained in a like part of the dividend. Now the cutting off directed in the rule is nothing more than taking like parts of both, the figures cut off from the dividend being (from the nature of Notation) the right hand figures of the remainder; wherefore the rule is evident. This useful mode of contraction saves a multitude of ciphers, and shortens the work amazingly.

34. Divide 1571234 by 20.

OPERATION.

2|0)157123|4(

Quotient $\cdot 78561\frac{14}{20}$

Proof 1571234

Explanation.

Here having cut off the cipher from the divisor, and the last figure 4 from the dividend, I divide the remaining figures in the latter by 2. Having divided the 3, I have 1 remaining, which is placed before the 4 cut off, making the remainder 14; this is placed over 20, the divisor, at the end of the quotient, as in the former rules. The proof arises by multiplying the quotient by the divisor 20, (Art. 30.) and adding in the remainder 14. For other proofs, divide by 4×5 and 5×4. Art. 39.

	35.	36.	37.
	3|00)100123|43(	4|000)791236|131(	5|0000)1234|5678(
Quotient	$\cdot 33374\frac{143}{300}$	$197809\frac{131}{4000}$	$\cdot 246\frac{45678}{50000}$
Proof	10012343	791236131	12345678

38. Divide 1234567 by 210.

OPERATIONS.

3)123456|7(	7)123456|7(4
7)$\cdot$41152 (6	3)$\cdot$17636 (2
$\cdot 5878\frac{187}{210}$	$\cdot 5878\frac{187}{210}$

Explanation.

Having cut off the cipher from the divisor, for the 21 I divide by 3×7 and 7×3, having first cut off the 7 from the dividend. For the remainder, in the first operation, I say 6 times 3 are 18, and place it before the 7, making 187. In the second operation, I say twice 7 are 14 and 4 are 18; this placed before the 7 cut off makes 187, which in both is placed over the divisor.

39. Divide 2468123 by 60. *Quotient* $41135\frac{23}{60}$.

40. Divide 12345200 by 700. *Quotient* 17636.

41. Divide 98765432 by 11000. *Quotient* $8978\frac{7432}{11000}$.

42. Divide 852768000 by 120. *Quotient* 7106400.

41. *When the divisor is any number greater than* 12.

RULE I. Having placed the divisor on the left as before, find how many times it is contained in the least number of figures possible of those on the left of the dividend, and set the quotient figure to the right.

II. Multiply the divisor by the quotient figure, and set the product under the forementioned left hand figures of the dividend.

III. Subtract the said product from the figures above it, and to the right of the remainder bring down the next figure (viz. the next to those already used) in the dividend, and find how often the divisor is contained in this number.

IV. Set the quotient figure to the right of the dividend as before; multiply the divisor by it; place the product under the

last found number; subtract, bring down, divide, &c. as before, until all the figures in the dividend are brought down[*].

---

[*] This rule differs from the operation directed in Art. 37 B. only in one particular, namely, *there* the work is for the most part performed mentally, whereas *here* the whole process is put down.

Learners usually experience some difficulty in finding how many times the divisor *will go* in the figures to be divided; the rule is, when the divisor is contained in its own number of figures, find how often the *first* figure of the divisor is contained in the *first* of those to be divided; but when the number of figures in the divisor is one less than the number of figures to be divided, find how often the *first* figure of the former will go in the *two first* of the latter. This method, if proper allowances be made for carrying, serves to determine the quotient figures very readily. Another method is, to multiply the divisor (previous to the operation) by 2, 3, 4, &c. to 9, and set the products in their order in a column under the divisor, and their respective multipliers opposite; then at every step of the work you have only to take that product which is nearest to, but not greater than, the number to be divided; place it under the said number, and put the opposite multiplier in the quotient. For example, to divide 42021687 by 23; thus,

```
1....23) 42021687(1827029
2....46| 23
3....69| ---
4....92| 190
5.. 115| 184
6.. 138| ---
7.. 161|   62
8.. 184|   46
9.. 207|   ---
       |   161
           161
           ---
             68
             46
             ---
             227
             207
             ---
              20
```

Here the left hand column contains the multipliers of the divisor; the second column contains the products, each opposite its proper multiplier; to the right of this stands the operation. First then, to find how often 23 goes in 42, I look among the products, and find that 23 is the only one not greater than 42; I therefore put 23 under the 42, and its opposite multiplier 1 in the quotient. Having subtracted, and brought down the 0, the number next to be divided is 190; the nearest number not greater than this among the products is 184; this I put below 190, and its multiplier 8 in the quotient: the next number which arises to be divided is 62; the nearest product for this is 46, which I put below it, putting the corresponding multiplier 2 in the quotient: I proceed in this manner until the work is finished.

The first method of proof is obvious; for it is plain, that if the dividend contain the divisor some number of times *exactly*, that number will be the quotient; and (this rule being the converse of Multiplication) the divisor multiplied by the quotient will produce the dividend *exactly* when the work is right. But if there be a remainder, it is plain that the dividend exceeds the aforementioned product by that remainder, which therefore must be added to the product in order to produce the dividend.

The second method by casting out the nines will be demonstrated algebraically in its proper place.

The third method of proof by adding, depends on this consideration, that the

Methods of proof.

I. Multiply the quotient and divisor together, add in the remainder, and the result will be like the dividend when the work is right.

II. Cast the nines out of the divisor and quotient, setting the remainders on opposite sides of a cross, as in Multiplication. Multiply these together, and, having cast out the nines from the product, set what is over at the top of the cross. Cast the nines out of the remainder; subtract what is over from the dividend, and cast the nines out of what remains; put the remainder at the bottom of the cross, and it will be like the top figure when the work is right.

III. Add the remainder, and all the lower lines of figures together, and if the work is right, the sum will be like the dividend.

43. Divide 550914 by 234.

OPERATION.

```
Divisor Dividend Quotient
 234 ) 550914 ( 2354                      Proof by Multiplication.
       468                                      Quotient 2354
       ---                                      Divisor   234
        829              Proof.                          ----
        702             \  0  /                          9416
       ----              \   /                          7062
       1271            0   X   5                       4708
       1170              /   \                  Rem.      78
       ----             /  0  \                       ------
        1014                                  Dividend 550914
         936
        ----
          78 Remainder
      ------
      550914 Proof by Addition.
```

Explanation.

First, I take as many (and no more) of the left hand figures of the dividend (viz. 550.) as contain the divisor 234; I then try how often 234 will go in 550,

---

product of the divisor and quotient, with the remainder added, equals the dividend; now the numbers here actually added are the remainder, and the products of the divisor into the several quotient figures, placed in order; and since the sum of the products of the parts into any number equals the product of the whole into that number, it follows that the sum of these products and the remainder will be equal to the dividend, if the operation be correct.

Another method of proof is, to subtract the remainder from the dividend, and divide the result by the quotient; the resulting quotient will be like the divisor when the work is right. Thus in ex. 43. if the remainder 78 be taken from the dividend, and the result 550836 be divided by the quotient 2354, the quotient of this division will be 234, which is equal to the given divisor.

and find it will go twice; I therefore put 2 in the quotient, (viz. on the right of the dividend,) and multiply the divisor 234 by it, the product 468 I then place under the 550. Next I subtract 468 from 550, and the remainder is 82; to this I bring down the 9, making 829; I try how often the divisor will go in this, and find it will go 3 times; I put 3 in the quotient, and multiply the divisor by it; the product 702 I place under the 829, and, subtracting the lower from the upper, the remainder is 127; I bring down 1 to this, and it becomes 1271; I try how often the divisor will go in this, and find it will go 5 times; I therefore put 5 in the quotient, and multiplying the divisor by it, I put the product 1170 under, and subtract it from 1271; to the remainder 101 I bring down the last figure in the dividend, viz. 4, making 1014; I find that the divisor is contained 4 times in this number; I put the 4 in the quotient, multiply the divisor by it, place the product under the 1014, and subtract as before; there being no more figures in the dividend to bring down, the 78 is the remainder.

In the proof by Multiplication, the quotient 2354 is multiplied by the divisor 234, and the remainder 78 added to the product, and the result being like the dividend shews that the work is right.

In the proof by Addition, I add the remainder and the lower lines of figures (not the upper) together, as they stand vertically, or under each other; thus 8 and 6 are 14; put down 4, and carry 1 to 7 are 8 and 3 are 11 and 0 are 11; put down 1, and carry 1 to the 9, 7, and 2, which make 19; put down 9, and carry 1 to the 1, 0, and 8, which make 10; put down 0, and carry 1 to the 1, 7, and 6, which make 15; put down 5, and carry 1 to the 4 is 5, which I put down.

In the proof by the cross, I cast the nines out of the divisor 234, and the 0 which remains I place on the left of the cross; I then cast the nines out of the quotient 2354, and place the remainder 5 on the right. I next multiply 5 and 0 together, and put the result 0 at the top. I then cast the nines out of the remainder 78, and, subtracting the 6 that is over from the dividend, I cast the nines out of the remainder, and the result is 0 like the top; wherefore the work is right.

44. Divide 304932 by 13.

OPERATION.

```
Divisor Dividend Quotient
   13 ) 304932 ( 23456
         26          13
        ---        -----
         44        70368
         39        23456
        ---            4 Rem.
          59       ------
          52       304932 Proof
         ---       ------
           73                      Proof.
           65                        8
          ---                      4 X 2
            82                       8
            78
           ---
             4 Rem.
        ------
        304932 Proof by Addition
```

45. Divide 1760598 by 23.

OPERATION.

```
Divisor. Dividend. Quotient.
   23 ) 1760598 ( 76547
        161          23
        ---       ------
         150      229641
         138      153094
         ---          17 Rem.
          125     -------
          115     1760598 Proof.
          ---
           109             Proof.
            92               1
           ---             5   2
            178              1
            161
            ---
             17 Rem.
        -------
        1760598 Proof by Addition.
```

46. Divide 123456789 by 9876.

OPERATION.

```
9876)123456789(12500           Proofs.
      9876                          12500
     -----              6            9876
      24696           3   8         -----
      19752             6           75000
      -----                        87500
       49447                      100000
       49380                     112500
       -----                          6789 Rem.
        6789 Rem.                ---------
     ---------                   123456789
     123456789 Proof.
```

47. Divide 382701 by 31. *Quot.* 12345. *rem.* 6.

48. Divide 814483 by 23. *Quot.* 35412. *rem.* 7.

49. Divide 24753819 by 26. *Quot.* 952069. *rem.* 25.

50. Divide 80132457 by 29. *Quot.* 2763188. *rem.* 5.

51. Divide 15185088 by 123. *Quot.* 123456.

52. Divide 85691764 by 243. *Quot.* 352641. *rem.* 1.

53. Divide 73671248 by 857. *Quot.* 85964. *rem.* 100.

54. Divide 175729358 by 3542. *Quot.* 49613. *rem.* 112.

55. Twelve men divide 20736 pounds equally among themselves; what is the share of each? *Ans.* 1728*l.*

56. A baker has 5138 pounds of dough, of which he intends to make loaves weighing 14 pounds each; how many will he have? *Ans.* 367.

57. In a certain frigate there are 176 common sailors, and

they share 30800l. prize-money equally; what sum does each receive? *Ans.* 175*l.*

58. How many times will a watch, which goes 31 hours, require winding up in 28458 hours? *Ans.* 918 *times.*

59. The crop of wheat on an estate is 1164 quarters; what quantity was sown, supposing every corn to have produced 97 on an average? *Ans.* 12 *quarters.*

60. How many times is the number 999 contained in the ten digits, arranged in their natural order? *Ans.* 1235803 *times, and* 693 *over.*

REDUCTION.

42. Reduction[b] teaches how to change numbers from one denomination to another, without altering their value; and is performed by multiplication and division.

When numbers are brought from a higher denomination into a lower, the operation is called Reduction *descending*, and is performed by *multiplication.*

When numbers are brought from a lower denomination into a higher, it is called Reduction *ascending*, and is performed by *division.*

43. *To bring great names into small, that is, to reduce numbers from a higher denomination to a lower.*

Rule. Multiply by the number denoting how many of the lower denomination make one of the higher.

Thus to bring pounds into shillings, I multiply the pounds by 20, because 20 shillings make one pound: to bring shillings into pence, I multiply the shillings by 12, because 12 pence make one shilling: and to bring pence into farthings, I multiply the pence by 4, because 4 farthings make one penny[c].

---

[b] From the Latin *reduco*, to restore, or bring back.

[c] The reason of this rule will be readily understood. Suppose it were required to reduce any number of pounds into farthings; the most convenient method would evidently be, by reducing the given number first into shillings, then into pence, and next into farthings. Now 1 pound contains (once twenty, or) twenty times 1 shilling; 2 pounds contain (twice twenty, or) 20 times 2 shillings; in like manner 3 pounds contain 20 times 3 shillings; and in general any number of pounds will contain 20 times that number of shillings. By similar reasoning it appears, that any number of shillings contains 12 times that number of pence; and any number of pence, 4 times that number of farthings. Wherefore, since there are 20 shillings in 1 pound, and 12 pence in 1 shilling, it follows, that there are 20 times 12, or 240 pence, in 1 pound; and likewise

44. *When there are intermediate denominations between the given denomination, and that to which you would reduce it.*

RULE. Reduce the given number step by step in order, through all the intermediate denominations, (by the foregoing rule,) until you have brought it down to the proposed denomination.

Thus to bring pounds into farthings, I first reduce the pounds into shillings, then the shillings into pence, and lastly the pence into farthings [d].

45. *When the given number consists of several denominations.*

RULE. Begin at the highest denomination, reduce it to the second, and to the result add the second denomination in the given number; reduce this sum to the third denomination, and to the result add the third denomination in the given number; proceed until you have arrived at the denomination required.

Thus to bring pounds, shillings, pence, and farthings, into farthings; I begin with the pounds, reduce them into shillings, and add the given shillings to the result; I then reduce this number into pence, and take in the given pence; and lastly I reduce this last number into farthings, and take in the given farthings.

46. *To bring small names into great; that is, to reduce numbers from a lower denomination to a higher.*

RULE. Divide by the number denoting how many of the lower denomination make one of the higher.

Thus to bring farthings into pence, I divide the farthings by 4, because 4 farthings make one penny; to bring pence into shillings, I divide the pence by 12, because 12 pence make one shilling; and to bring shillings into pounds, I divide the shillings by 20, because 20 shillings make one pound [e].

(since 4 farthings make 1 penny) there will be 20 times 12 times 4, or 960 farthings, in 1 pound; consequently any number of pounds multiplied by 240 will produce the number of pence, and by 960, the number of farthings, in those pounds.

[d] When there are shillings, pence, or farthings, connected with the given number, it is plain that these must be added in successively, each with its like; viz. shillings with shillings, pence with pence, and farthings with farthings. This being understood, the reason of the rule, as applied to weights and measures, will likewise be evident.

[e] Because there are 4 farthings in 1 penny, 8 farthings in 2 pence, 12 farthings in 3 pence, and in general 4 times the number of farthings in any number of pence; it follows that there will be in any number of farthings one

47. *When there are intermediate denominations between the given one, and that to which it is required to be reduced.*

RULE. Reduce the given number step by step in order from the given denomination upward, through all the intermediate ones, until you have brought it to the proposed denomination.

Thus to bring farthings into pounds, I first reduce them to pence, then the pence to shillings, and then the shillings to pounds.

When there is a remainder, it is of the same denomination with the dividend from whence it arises [f].

MONEY [g].

4 farthings (q.) *make*	1 penny, d.	q. 4 = 1 d.
12 pence	1 shilling, s.	48 = 12 = 1 s.
5 shillings	1 crown, cr.	240 = 60 = 5 = 1 cr.
20 shillings	1 pound, L.	960 = 240 = 20 = 4 = 1 L.
21 shillings	1 guinea, *guin*.	
27 shillings	1 moidore, *moid*.	

$\frac{1}{4}$ denotes one farthing: $\frac{1}{2}$ two farthings, or one halfpenny: and $\frac{3}{4}$ three farthings.

EXAMPLES.

1. In 53l. how many farthings?

OPERATION.

$$\begin{array}{rl} L. & \\ 53 & \\ 20 & \\ \hline 1060 & \textit{shillings}. \\ 12 & \\ \hline 12720 & \textit{pence}. \\ 4 & \\ \hline \textit{Ans.}\ 50880 & \textit{farthings}. \end{array}$$

Explanation.

Here we have to reduce great into small, consequently multiplication must be used. I first multiply 53 by 20, which produces shillings; these I multiply by 12, which produces pence; and these I multiply by 4, which produces farthings, or what was required in the question. This operation is proved by that of the following example; they mutually prove each other, and may serve as a pattern to shew how other examples in this rule are proved.

---

fourth that number of pence; in like manner, in any number of pence there will be one twelfth that number of shillings; and in any number of shillings one twentieth that number of pounds. This rule is therefore evident, as being the converse of the former rule.

[f] This will be plain, from the consideration that every remainder, being a part of the dividend, is evidently of the same name with it.

[g] Here, as well as in each of the weights and measures, there are two tables, the first of which is mostly used in reduction; the second shews what number of *every inferior denomination* is contained in each superior one.

Pounds, shillings, pence, and farthings, are usually denoted by the Latin initials L. s. d. q. L denoting *libra*, a pound; s, *solidus*, a shilling; d, *denarius*, a penny; and q, *quadrans*, a farthing.

2. In 50880 farthings, how many pounds?

OPERATION.

farthings.
4)50880(
12)12720(*pence.*
2|0) 106|0(*shillings.*
Ans. 53 *pounds.*

Explanation.

Here we bring small into great, and therefore divide. I divide the farthings by 4, which produces pence; the pence by 12, which gives shillings; and the shillings by 20, which gives pounds, or the answer required.

3. How many farthings are there in 23*l.* 14*s.* 5*d.*¼?

OPERATION.

L. *s.* *d.*
23 14 5¼
20
474 *shillings.*
12
5693 *pence.*
4
Ans. 22773 *farthings.*

Explanation.

I multiply the pounds, viz. 23, by 20, and to the product I add the 14 shillings, taking in the 4 in the place of units, and 1 in the place of tens. Next I multiply 474 shillings by 12, and take in the 5 pence to the product. I then multiply 5693 pence by 4, and to the product take in the 1 farthing, and the result is the answer required.

4. Bring 45546 farthings into pounds.

OPERATION.

farthings.
4)45546(2
12)11386(10
2|0) 94|8
Ans. 47*l.* 8*s.* 10*d.*½

Explanation.

I divide by 4, 12, and 20, as before, (Ex. 2.) The 2 remainder after the first division are 2 farthings, or ½*d.* the 10 remainder after the second are pence, and the 8 cut off in the third division are shillings. These are collected with the last quotient, making together 47*l.* 8*s.* 10*d.*½ the answer.

5. In 123*l.* how many pence? *Ans.* 29520.
Multiply by 20 *and by* 12.

6. In 456*l.* how many farthings? *Ans.* 437760.
Multiply by 20, 12, *and* 4.

7. In 59040 pence, how many pounds? *Ans.* 246.
Divide by 12 *and* 20.

8. In 266880 farthings, how many pounds? *Ans.* 278.
Divide by 4, 12, *and* 20.

9. In 345 crowns, how many farthings? *Ans.* 82800.
Multiply by 5, 12, *and* 4.

10. In 165600 farthings, how many crowns? *Ans.* 690.
Divide by 4, 12, *and* 5.

11. In 487 halfcrowns, how many halfpence? *Ans.* 29220.
Multiply by 30 *and* 2.

12. In 97440 halfpence, how many halfcrowns? *Ans.* 1624.
Divide by 2 *and* 30.

13. In 243 guineas, how many farthings? *Ans.* 244944.
Multiply by 21, 12, *and* 4.

14. In 734832 farthings, how many guineas? *Ans.* 729.
Divide by 4, 12, *and* 21.

15. In 157 moidores, how many halfpence? *Ans.* 101736.
Multiply by 27, 12, *and* 2.

16. In 610416 farthings, how many moidores? *Ans.* 471.
Divide by 4, 12, *and* 27.

17. In 4*l.* 3*s.* 2*d.*$\frac{1}{4}$, how many farthings? *Ans.* 3993.
See Example 3.

18. In 11979 farthings, how many pounds? *Ans.* 12*l.* 9*s.* 6*d.*$\frac{3}{4}$
See Example 4.

19. In 5*l.* 12*s.* 3*d.*$\frac{1}{2}$, how many farthings? *Ans.* 5390.

20. In 16170 farthings, how many pounds? *Ans.* 16*l.* 16*s.* 10*d.*$\frac{1}{2}$

21. In 8*l.* 15*s.* 4*d.*$\frac{1}{2}$, how many halfpence? *Ans.* 4209.

22. In 12627 halfpence, how many pounds? *Ans.* 26*l.* 6*s.* 1*d.*$\frac{1}{2}$

23. In one thousand guineas, how many farthings? *Ans.* 1008000.

24. In ten thousand farthings, how many crowns? *Ans.* 41 *crowns, and* 3*s.* 4*d.* *over.*

48. Sometimes it is necessary to reduce numbers from one denomination to another, such, that there is no number of the one contained exactly in one of the other: operations of this kind require both multiplication and division, and are therefore called Reduction ascending and descending.

RULE. Having considered what denomination is given, and what is required, reduce the given one to some inferior denomination common to them both, that is, to one which is contained some number of times exactly in each of them, by Art. 43. then reduce it from this into the denomination required[h], by Art. 46.

---

[h] The truth of this method will be plain from the preceding notes.

25. Reduce 54120 guineas into pounds.

OPERATION.

guin.
54120
21
―――
54120
108240
$2|0)113652|0$
Ans. 56826 *pounds.*

Explanation.

I find that a shilling is the greatest denomination contained some number of times without remainder in both a guinea and a pound; I therefore bring 54120 guineas into shillings by multiplying by 21; (Art. 43.) and lastly, I bring the shillings into pounds by dividing by 20, (Art. 46.)

26. How many half guineas are there in 1234 crowns?

OPERATION.

cr.
1234
10
―――
21)12340(587 *half guin. Ans.*
105
―――
184
168
―――
160
147
―――
13 *sixpences, or 6s. 6d. over.*

Explanation.

Here the greatest denomination common to half a guinea and a crown is 6d.; there are 10 sixpences in a crown, and 21 in half a guinea; I therefore reduce the crowns into sixpences by multiplying by 10, (Art. 43.) and the sixpences into half guineas by dividing by 21 (Art. 46.); the answer is 587 half guineas, and 13 sixpences, or 6s. 6d. over.

27. In 2142 pounds, how many guineas? *Ans.* 2040.
Multiply by 20, and divide by 21.

28. In 3420 guineas, how many half crowns? *Ans.* 28728.
Multiply by 42, and divide by 5.

29. In 840 moidores, how many guineas? *Ans.* 1080.
Multiply by 27, and divide by 21.

30. In 307*l.* 16*s.* how many moidores? *Ans.* 228.

31. In 57015 crowns, how many guineas? *Ans.* 13575.

32. In 10500 moidores, how many pounds? *Ans.* 14175.

TROY WEIGHT[b].

49. Troy weight is used for weighing such commodities as are of a pure nature, and very little subject to waste, as gold,

[b] Troy weight (called in the old books *Trone weight*) is supposed to have originated in France, and to have taken its name from *Troyes*, a considerable city in the department of Aube.

The origin of all English weights was a corn of sound ripe wheat taken out of the middle of the ear; 32 of these well dried were to make 1 pennyweight, 20 pennyweights an ounce, and 12 ounces a pound, according to 51 Henry III. 31 Edward I. and 12 Henry VII.; but afterwards the penny

silver, and jewels; it is likewise employed to determine the comparative strength of liquors.

24 grains[i] (*gr.*) make	1 pennyweight (*dwt.*)	*gr.* 24 = 1*dwt.*
20 pennyweights . .	1 ounce. (*oz.*)	480 = 20 = 1*oz.*
12 ounces	1 pound. (*lb.*)	5760 = 240 = 12 = 1*lb.*

33. In 123*lb.* 4*oz.* 5*dwt.* 6*gr.* how many grains?

OPERATION.

```
lb.  oz. dwt. gr.
123   4   5   6
 12
1480 ounces.
  20
29605 pennywts.
   24
118426
59210
Ans. 710526 grains.
```

Explanation.

I begin at the highest denomination (*lb.*), and multiply by 12, taking in the 4; this gives ounces. The ounces I multiply by 20, and take in the 5 for pennyweights. The pennyweights I multiply by 24, and take in the 6 for grains, which gives the answer.

34. In 39483 grains, how many pounds?

OPERATION.

```
gr.         2|0)
24 ) 39483 ( 164|5
     24   12) 82 (
     154   Ans. 6lb. 10oz. 5dwt. 3gr.
     144
      108
       96
       123
       120
         3 grains.
```

Explanation.

I divide grains by 24, and the quotient 1645 is dwts. these I divide by 20, and the quotient is ounces; these I divide by 12, and the quotient is pounds. The 3 remaining after the first division are grains, the 5 cut off in the second are dwts. the 10 over in the third are ounces; these placed in order give 6*lb.* 10*oz.* 5*dwt.* 3*gr.*

35. In 6*lb.* 10*oz.* 5*dwt.* 3*gr.* how many grains? *Ans.* 39483.

36. In 710526 grains, how many pounds? *Ans.* 123*lb.* 4*oz.* 5*dwt.* 6*gr.*

---

weight was divided into 24 parts, called *grains*, as at present. This seems to have been the only legal weight used in England from the Norman conquest to the year 1533, when an act was passed, authorizing the use of avoirdupois weight, by which meat was to be bought and sold. *Rapin's Hist. of Eng.* vol. vi.

[i] The grain troy is thus divided and subdivided by the moneyers; viz. a grain into 20 mites, a mite into 24 droits, a droit into 20 periots, and a periot into 24 blanks.

37. Reduce 175*lb.* into grains. *Ans.* 1008000.

38. How many pounds are there in 201600 grains? *Ans.* 35.

APOTHECARIES' WEIGHT.

50. The Apothecaries compound their medicines by this weight, but never use it for buying or selling: the pound and ounce are exactly the same as the pound and ounce troy; but the smaller divisions in this weight have the advantage in point of convenience, where great accuracy is required [k].

		gr.	℈	ʒ	℥	℔
20 grains (*gr.*) *make*	1 scruple. ℈.	20 =	1			
3 scruples	1 dram. ʒ.	60 =	3 =	1		
8 drams	1 ounce. ℥.	480 =	24 =	8 =	1	
12 ounces	1 pound. ℔.	5760 =	288 =	96 =	12 =	1

39. In 10℔ 11℥ 7ʒ 2℈ 19*gr.* how many grains?

OPERATION.

℔	℥	ʒ	℈	*gr.*
10	11	7	2	19

12

131 *ounces.*

8

1055 *drams.*

3

3167 *scruples.*

20

Ans. 63359 *grains.*

Explanation.

Beginning at the pounds, I multiply by 12, and take in the 11 for ounces; these I multiply by 8, and take in the 7 for drams; the drams I multiply by 3, and take in the 2 for scruples; which latter I multiply by 20, and take in the 19 for grains.

---

[k] Drugs of every description are bought and sold by Avoirdupois weight.

Physicians, as well as their learned brethren the lawyers, conceal their secret from the vulgar under the disguise of abbreviated Latin; a practice derived from the Schoolmen: thus, in their prescriptions,

℞ denotes *recipe*, or take.
a, áá, or *ana*, ... of each the same quantity.
ss. .... *semis*, half of any thing.
coch. .... *cochleare*, a spoonful.
cong. .... *congius*, a gallon.
P. .... *pugil*, as much as can be held between the thumb and two fingers.
M. .... *manipulus*, a handful.
q. s. .... *quantum sufficit*, a sufficient quantity.
℈j .... a scruple.
ʒj or ℈iij .... a dram.
℥j or ʒviij. ...... an ounce.
℔j or ℥xij......... a pound.

40. In 159024 grains, how many pounds?

OPERATION.

```
   gr.
2|0(15902|4(
  3)7951(1
  8)2650(2
 12)331(
Ans. 27℔ 7℥ 2ʒ 1℈ 4gr.
```

Explanation.

I divide grains by 20 for scruples, scruples by 3 for drams, drams by 8 for ounces, and ounces by 12 for pounds. There are 4 over in grains, 1 in scruples, 2 in drams, and 7 in ounces; these placed in order after the pounds give the answer.

41. In 63359 grains, how many pounds? *Ans.* 10℔ 11℥ 7ʒ 2℈ 19*gr.*

42. In 27℔ 7℥ 2ʒ 1℈ 4*gr.* how many grains? *Ans.* 159024.

AVOIRDUPOIS WEIGHT.

51. Avoirdupois weight is used for almost all kinds of merchandise; it seems to have been adopted for the purpose of giving sufficient weight in selling commodities of a coarse and drossy nature, a pound in this weight being equal to 14*oz.* 11*dwt.* 16*gr.* troy nearly[1].

16 drams (*dr.*)	*make*	1 ounce. *oz.*
16 ounces	. . . .	1 pound. *lb.*
28 pounds	. . . .	1 quarter. *qr.*
4 quarters	. . . .	1 hundred weight. *cwt.*
20 hundred weight	. .	1 ton. *t.*

$$
\begin{array}{r}
dr. \\
16 = 1oz. \\
256 = 16 = 1lb. \\
7168 = 448 = 28 = 1qr. \\
28672 = 1792 = 112 = 4 = 1cwt. \\
573440 = 35840 = 2240 = 80 = 20 = 1ton.
\end{array}
$$

[1] Avoirdupois weight began to be used for butchers' meat only about the end of the 15th century, and became a legal weight for that purpose by the 24 Henry VIII. by degrees its use was extended to all kinds of coarse and heavy merchandise, so that now all kinds of grocery and chandlery wares, meat, bread, corn, tallow, pitch, tar, turpentine, iron, brass, copper, lead, tin, &c. &c. are weighed by Avoirdupois weight.

A pound Troy equals 13*oz.* $2\frac{3}{4}$*dr.* nearly Avoirdupois, and an ounce Troy is about 1*oz.* $1\frac{1}{7}$*dr.* Avoirdupois: also, an ounce Avoirdupois equals 18*dwts.* $5\frac{1}{2}$*gr.* Troy; 175 pounds Troy = 144 pounds Avoirdupois; 175 ounces Troy = 192 ounces Avoirdupois; consequently the Troy pound is less, and the ounce greater, than the pound and ounce Avoirdupois.

The name *Avoirdupois* is derived from the French *avoir*, to have, and *du poids*, weight.

Some commodities for which this weight is used have denominations peculiar to themselves; as the following.

Bread and Flour.	*lb.*	*oz.*
A peck loaf *weighs*	17	6
Half peck	8	11
Quartern	4	5½
Half quartern	2	2¾
A peck of flour	14	0
A bushel	56	0
A sack, or 5 bushels,	280	0

Butter, Cheese, and Meat.	
8 pounds *are*	a clove.
2 cloves	a stone.
56 pounds	a firkin of butter.
224 pounds . . .	a barrel.
32 cloves	a wey in Suffolk.
42 cloves	a wey in Essex.
8 pounds	a stone of butcher's meat.

Wool.	
14 pounds *are*	a stone.
2 stone	a tod.
6½ tod	a wey.
2 weys	a sack.
12 sacks	a last.

Hay and Straw.	
36 pounds *are*	a truss of straw.
56 pounds	a truss of old hay.
60 pounds	a truss of new hay.
36 trusses	a load.

Iron, Steel, and Lead.	
14 pounds *are*	a stone.
120 pounds . . .	a faggot.
19½ cwt.	a fother.

43. In 1*t.* 2*cwt.* 3*qr.* 4*lb.* how many pounds?

OPERATION.

```
  t. cwt. qr. lb.
  1  2   3   4
  20
  --
  22 cwt.
   4
  --
  91 qr.
  28
  ---
  732
 182
 ----
Ans. 2552 lb.
```

Explanation.

I multiply the ton by 20, and take in the 2, which gives cwts. these I multiply by 4, and take in the 3, which gives quarters; these I multiply by 28, and take in the 4 for pounds.

44. In 293575 drams, how many cwt.?

OPERATION.

```
dr.       16)     28)    4)
16)293575(18348(1146 ( 40
   16        16    112   10cwt. 0qr. 26lb. 12oz. 7dr. Ans.
   133       23     26 lb.
   128       16
     55      74
     48      64
     77     108
     64      96
     135     12 ounces.
     128
       7 drams.
```

Explanation.

In the first operation, I divide drams by 16, and the quotient is 18348 ounces; these I divide by 16, and the quotient is 1146 lbs.; these I divide by 28, and the quotient is 40 quarters; these I divide by 4, and the quotient is 10 cwt.; this, with 0 over in quarters, 26 in pounds, 12 in ounces, and 7 in drams, constitutes the answer.

In the second operation, instead of the two sixteens, I divide twice, by 4 × 4, and instead of 28, by 7 × 4, which makes short division of it: the remainders are calculated by Art. 39. in Division.

Or thus.

dr.

$16\left\{\begin{array}{l|l|l} 4 & 293575 & 3 \\ 4 & 73393 & 1 \end{array}\right\}$ 7 *dr.*

$16\left\{\begin{array}{l|l|l} 4 & 18348 & 0 \\ 4 & 4587 & 3 \end{array}\right\}$ 12 *oz.*

$28\left\{\begin{array}{l|l|l} 7 & 1146 & 5 \\ 4 & 163 & 3 \end{array}\right\}$ 26 *lbs.*

4| 40|

Ans. 10*cwt.* 0*qr.* 26*lbs.* 12*oz.* 7*dr.*

45. In 32*cwt.* how many pounds? *Ans.* 3584.
See Example 43.

46. In 10752 pounds, how many cwt.? *Ans.* 96.
Divide by 28 *and* 4.

47. In 14*cwt.* how many drams? *Ans.* 401408.
Multiply by 4, 28, 16, *and* 16.

48. In 200704 drams, how many cwt. *Ans.* 7.
Divide by 16, 16, 28, *and* 4.

49. In 2*cwt.* 3*qr.* 4*lb.* 5*oz.* 6*dr.* how many drams? *Ans.* 79958.

50. In 11*t.* 12*cwt.* 3*qr.* 14*lb.* how many ounces? *Ans.* 417312.

51. In 12760*lb.* how many tons? *Ans.* 5*t.* 13*cwt.* 3*qr.* 20*lb.*

52. In 4*t.* 3*cwt.* 2*qr.* 1*lb.* how many drams? *Ans.* 2394368.

52. LONG MEASURE.

In estimating the contents of magnitudes, different measures are employed; some are measured by their length only, some by their length and breadth conjointly, and some by their length, breadth, and thickness conjointly.

Long Measure[m] is that which measures any one of these dimensions, being applied to distance only.

4 lines (*l.*) *make*	1 barley-corn. *bc.*
3 barley-corns, or 12 lines .	1 inch. *in.*
4 inches	1 hand[n]. *h.*
9 inches	1 span. *sp.*
12 inches	1 foot[o]. *f.*
18 inches	1 cubit[p]. *cub.*
3 feet	1 yard[q]. *yd.*
6 feet	1 fathom[r]. *fath.*
$5\frac{1}{2}$ yards	1 rod, pole, or perch[s]. *r* or *p.*
40 poles	1 furlong. *fur.*
8 furlongs	1 mile. *m.*
3 miles	1 league.

---

[m] Long Measure, which is that from whence all the other measures are derived, owes its origin to the length of a grain of barley: three grains of sound ripe barley being taken out of the middle of the ear, well dried, and laid end to end in a row, were considered as an inch, which in this measure is called *the measuring unit.* As the length of the barley-corn cannot be fixed, so the inch according to this method will be uncertain; but to remedy this inconvenience, there are *standard* measures as well as weights kept in the Exchequer chamber, Guildhall, for the purpose of comparing the weights and measures used by dealers; this helps to secure the public from imposition and fraud, as the use of measures or weights less than the *standard* is prohibited by law.

[n] The *hand* is used in measuring the height of horses.

[o] The foot is supposed to be taken from the length of the human foot.

[p] The cubit is a measure used by the ancients, and often mentioned in their writings; it is supposed to have been originally taken from the length of that part of a man's arm between the point of the elbow and the extremity of the hand.

[q] The yard is said to have been taken from the arm of King Henry I. in 1101.

[r] The fathom is taken from the utmost extent of both arms, when stretched into a right line; it is applied to measuring mines, wells, pits, and depths, in general; also the length of ropes, &c.

[s] The length of the pole differs in different parts of England; in the neighbourhood of London it is $5\frac{1}{2}$ yards; in some counties it is 6, in Lancashire 7, and in Cheshire 8 yards.

lines.
12= 1 inch.
144= 12 = 1 foot.
432= 36 = 3 = 1 yard.
2376= 198 = $16\frac{1}{2}$ = $5\frac{1}{2}$ = 1 pole.
95040= 7920 = 660 = 220 = 40 = 1 furlong.
760320= 63360 = 5280 =1760=320= 8 =1 mile.
2280960=190080=15840=5280=960=24=3=1 league.

53. In 12*m.* 3*fur.* 4*p.* 1*yd.* how many yards?

OPERATION.

m.	*fur.*	*p.*	*yd.*
12	3	4	1

8
99 *furlongs.*
40
3964 *poles.*
$5\frac{1}{2}$
19821
1982
Ans. 21803 *yards.*

Explanation.

The miles are here multiplied by 8, (because 8 furlongs make a mile,) and the 3 furlongs are added to the product, this gives furlongs; these are multiplied by 40, (because 40 poles make a furlong,) and the 4 taken in, this gives poles; these are multiplied by $5\frac{1}{2}$, (because $5\frac{1}{2}$ yards make a pole,) and the 1 is taken in to the product, this gives yards.

To multiply 3964 by $5\frac{1}{2}$, I first multiply it by 5, then divide it by 2, and lastly I add the product and quotient together.

54. In 96800 yards, how many miles?

OPERATION.

yards. 8)
22|0)9680|0 (440
88 *Ans.* 55 *miles.*
88
88
0

Explanation.

The division by $5\frac{1}{2}$ cannot be performed conveniently without fractions; I therefore divide by 220, (because 220 yards make a furlong,) this gives furlongs; these I divide by 8, (because 8 furlongs make a mile,) and the result is miles.

If you wish to bring the yards into poles, multiply by 2, and divide the product by 11; then dividing successively by 40 and by 8, will produce the very same answer.

55. In 50 leagues, how many yards? *Ans.* 264000.
Multiply by 3, 8, 40, *and* $5\frac{1}{2}$.

56. In 2661120 barley-corns, how many miles? *Ans.* 14.
Divide by 3, 12, 3, 220, *and* 8.

57. In 1*m.* 2*fur.* 3*p.* how many yards? *Ans.* $2216\frac{1}{2}$.

58. In 4755801600 barley-corns, how many leagues? *Ans.* 8340.

59. In 7 miles, how many lines? *Ans.* 5322240.

53. CLOTH MEASURE.

$2\frac{1}{4}$ inches *make*	1 nail. *n.*
4 nails	1 quarter. *qr.*
3 quarters	1 Flemish ell. *F. E.*
4 quarters	1 yard*. *yd.*
5 quarters	1 English ell. *E. E.*
6 quarters	1 French ell. *Fr. E.*

60. In 25*yds.* 3*qr.* 2*n.* how many nails? *Ans.* 414.
Multiply the 25 *by* 4, *and take in* 3; *multiply the result by* 4, *and take in* 2.

61. In 1720 nails, how many English ells? *Ans.* 86.
Divide by 4 *and* 5.

62. In 137 Flemish ells, how many nails? *Ans.* 1644.
Multiply by 3 *and* 4.

63. In 3456 nails, how many French ells? *Ans.* 144.
Divide by 4 *and* 6.

64. In 1264 yards, how many Flemish ells? *Ans.* 1685 *F.E.* 1 *qr.*
Multiply by 4, *and divide by* 3.

65. In 1792 Flemish ells, how many yards? *Ans.* 1344.
Multiply by 3, *and divide by* 4.

66. In 480 English ells, how many Flemish ells? *Ans.* 800.

67. In 12300 yards, how many English ells? *Ans.* 9840.

68. 52*E.E.* 2*qr.* 3*n.* how many nails? *Ans.* 1051.

SQUARE OR SUPERFICIAL MEASURE.

54. Square Measure is used to measure surfaces in which both length and breadth are estimated, as land, flooring, roofing, walling, wainscotting, plastering, painting, &c.

The smallest measure (called the *measuring unit*) here used is a square inch, or a little square, every side of which is an inch in length*.

---

* As in Long Measure a line of an inch in length was taken for *the measuring unit*, so here a square surface of an inch in length, and consequently the same in breadth, is assumed as *the measuring unit*; and as many times as this little square is contained in any superficial space, so many square inches is that space said to consist of: thus a square foot contains 144 square inches; for if the sides of a square of a foot each way be divided each into 12 equal parts, each part will be an inch; and if the opposite divisions be joined, the square foot will be divided into 144 equal squares, each a square inch. Hence a square foot

144 square inches *make* 1 square foot.
9 square feet . . . . . . 1 square yard.
$30\frac{1}{4}$ square yards . . . . 1 square pole, or perch.
40 square poles . . . . . 1 rood. *r.*
4 roods . . . . 1 acre. *a.*
640 acres . . . . 1 square mile.

inches.				
144=	1 foot.			
1296=	9 =	1 yard.		
39204=	$272\frac{1}{4}$=	$30\frac{1}{4}$ =	1 pole.	
1568160=	10890=	1210=	40 =	1 rood.
6272640=	43560=	4840=	160=	4=1 acre.

69. In 12 acres, how many square yards?

OPERATION.

```
     a.
     12
      4
     48   roods.
     40
   1920   poles.
   30¼
  57600
    480
Ans. 58080 yards.
```

Explanation.

Multiplying acres by 4 produces roods; roods by 40 produces poles; poles by $30\frac{1}{4}$ produces yards. The multiplication by $30\frac{1}{4}$ is thus performed; I first multiply 1920 by 30, then I divide it by 4 for the $\frac{1}{4}$. I place the latter result under the former, and add them together.

70. In 12345 square poles, how many acres?

OPERATION.

```
         p.
4|0)1234|5(25
  4)308
Ans. 77a. 0r. 25p.
```

Explanation.

Here I divide poles by 40 for roods, and roods by 4 for acres. The 2 over in perches together with the 5 cut off make 25 perches over.

71. In 27*a.* 3*r.* 35*p.* how many perches? *Ans.* 4475.

72. In 4637 perches, how many acres? *Ans.* 28*a.* 3*r.* 37*p.*

73. In 25 square yards, how many square inches? *Ans.* 32400.

$= 12 \times 12$ inches; a square yard $= 9 \times 9$ square feet; a square pole $= 5\frac{1}{2} \times 5\frac{1}{2}$ square yards, &c.

Land is usually measured by a chain invented by the Rev. Edmund Gunter, Professor of Astronomy at Gresham College, and therefore called Gunter's chain; this chain is 66 feet long, and consists of 100 equal links. Ten chains in length, and one in breadth, (or $100 \times 100 \times 10 = 100000$ links) make one acre.

CUBIC OR SOLID MEASURE.

55. Cubic Measure is used for measuring solid bodies, in which length, breadth, and thickness, are estimated.

The smallest measure here used, and which is therefore called the *measuring unit*, is a cubic inch, or a solid in the shape of dice, of an inch long, an inch wide, and an inch in depth [t].

1728 cubic inches	*make*	1 cubic foot.
27 cubic feet	....	1 cubic yard.
40 cubic feet of rough timber	}	1 ton or load.
50 cubic feet of hewn timber	}	1 ton or load.
42 cubic feet	....	1 ton of shipping.
108 cubic feet	....	1 stack of billets.
128 cubic feet	....	1 cord of ditto.

74. In 120 cubic yards, how many cubic inches ?

OPERATION.

```
  yards.
   120
    27
  ----
  3240 cubic feet.
  1728
 -----
 25920
 6480
22680
3240
-------
Ans. 5598720 cubic inches.
```

Explanation.

Here I multiply yards by 27, which produces feet, and feet by 1728, which produces inches.

[t] Every thing in nature to which measuring can be applied has *three* dimensions, namely, *length*, *breadth*, and *thickness*: but for the sake of convenience some things are considered of one dimension only, viz. length; such as distances of places, ropes, cloths, tapes, ribbon, &c. to these Long Measure is applied: others are considered as of two dimensions, viz. length and breadth; as land, painting, paving, &c. these are estimated by Square Measure: but there are many things which require to be estimated by all three of their dimensions, such as timber, stone, the capacity of vessels, &c. these are the subject of Cubic Measure. If each side of a cube (which is a solid contained by 6 equal squares) be 12 inches long, and divided into inches, and if the similar divisions of the opposite sides be joined by straight lines, (as has been shewn in the preceding note,) the given cube will be divided into 1728 equal cubes, each a cubic inch. Hence a cubic foot $= 12 \times 12 \times 12$ cubic inches; a cubic yard $= 3 \times 3 \times 3$ cubic feet, &c.

75. In 2799360 cubic inches, how many cubic yards? *Ans.* 60.
Divide by 1728, *and* 27.

76. How many cubic inches are there in a load of rough timber? *Ans.* 69120.

77. How many bales of cotton of a cubic yard each can be stowed in a ship of 100 tons burthen? *Ans.* 155, *and* 15 *over*.

WINE MEASURE.

56. Wines, spirits, cider, perry, mead, vinegar, oil, and milk, are sold by this measure[v].

4 gills	*make*	1 pint. *pt.*
2 pints	....	1 quart. *qt.*
4 quarts	....	1 gallon. *gal.*
10 gallons	....	1 anker. *a.*
18 gallons	....	1 rundlet. *run.*
42 gallons	....	1 tierce. *tier.*
63 gallons	....	1 hogshead. *hhd.*
2 tierces	....	1 puncheon. *pun.*
2 hogsheads	....	1 pipe. *p.*
2 pipes	....	1 tun. *t.*

---

[v] The measure for different kinds of wine differs considerably, the pipe containing from 110 to 140 gallons: hence it is usual for dealers to charge for what the pipe contains, which is found by actual guaging.

A statute made in the reign of Henry III. ordained, that the wine gallon should contain eight pounds troy of wheat taken from the middle of the ear, and well dried. During many ages Wine Measure was the only measure sanctioned by law for measuring any commodity whatever; at length other measures were introduced for less pure liquors, and for dry goods.

A law was made in the reign of Queen Anne, whereby the wine gallon is required to measure 231 cubic inches; consequently a pint measures $28\frac{7}{8}$ cubic inches.

A tun of distilled water weighs 18cwt. avoirdupois, and a tun of wine about 18cwt. 3qrs.; consequently a pint of wine will weigh about 16oz. $9\frac{1}{7}$ drams.

pints.
2= 1 quart.
8= 4 = 1 gallon.
336= 168= 42=1 tierce.
504= 252= 63=1½=1 hogshead.
672= 336= 84=2 =1⅓=1 puncheon.
1008= 504=126=3 =2 =1½=1 pipe.
2016=1008=252=6 =4 =3 =2=1 tun.

78 In 13*t.* 1*p.* 1*hhd.* 12*gall.* of wine, how many pints?

OPERATION.

```
     t.  p. hhd. gall.
     13  1   1   12
      2
     --
     27 pipes
      2
     --
     55 hogsheads.
     63
    ---
    167
   331
   ----
   3477 gallons.
      4
  -----
  13908 quarts.
      2
  -----
Ans. 27816 pints.
```

Explanation.

In multiplying by 63 to reduce hogsheads into gallons, I have 12 gallons to take in; the 2 is taken in in the first place of the multiplication by 3, and the 1 in the first place of the multiplication by 6. The rest is obvious.

79. In 126720 gills, how many ankers?

```
      gills.
   4)126720
   2)31680 pints.
   4)15840 quarts.
  10)3960 gallons.
Ans. 396 ankers.
```

80. In 5*hhds.* 43*gall.* 2*qt.* 1*pt.* how many pints? *Ans.* 2869.

81. In 5738 quarts, how many tuns? *Ans.* 5*t.* 1*p.* 48*gall.* 2*qt.*

BEER MEASURE.

57. This measure is used for ale, strong-beer, and small [*].

2 pints	*make*	1 quart.
4 quarts	....	1 gallon.
9 gallons	....	1 firkin. *fir.*
2 firkins	....	1 kilderkin. *k.*
2 kilderkins	..	1 barrel. *bar.*
$1\frac{1}{2}$ barrel	....	1 hogshead.
2 hogsheads	..	1 butt.

pints.
2 = 1 quart.
8 = 4 = 1 gallon.
72 = 36 = 9 = 1 firkin.
144 = 72 = 18 = 2 = 1 kilderkin.
288 = 144 = 36 = 4 = 2 = 1 barrel.
432 = 216 = 54 = 6 = 3 = $1\frac{1}{2}$ = 1 hogshead.
864 = 432 = 108 = 12 = 6 = 3 = 2 = 1 butt.

82. In 14 hogsheads of beer, how many quarts? *Ans.* 3024.
Multiply by $1\frac{1}{2}$, 2, 2, 9, *and* 4.

83. In 12345 pints of ale, how many kilderkins? *Ans.* 85*k.* 1*fir.* 4*gal.* 1*pt.* *Divide by* 2, 4, 9, *and* 2.

84. In 216 barrels of ale, how many hogsheads? *Ans.* 144.
Multiply by 2, *and divide by* 3.

85. In 216 hogsheads of beer, how many barrels? *Ans.* 324.

---

[*] Formerly the ale firkin consisted of $8\frac{1}{2}$ gallons, and the beer firkin of 9. A late writer has shewn, that this distinction is not at present attended to, but that the firkin, whether of ale or beer, contains 9 gallons. A gallon beer measure contains 282 cubic inches; consequently a pint contains $35\frac{1}{4}$ cubic inches. The wine gallon bears nearly the same proportion to the beer gallon, that a pound troy does to a pound avoirdupois; for the troy pound is to the avoirdupois pound as 144 to 175; and the wine gallon is to the beer gallon as 231 to 282; which is nearly the same.

A cubic foot of water weighs 1000 ounces avoirdupois, and a cubic foot of beer 1028 ounces; consequently a butt of beer will weigh about $17880\frac{1}{2}$ ounces, or 9*cwt.* 3*qr.* 25*lb.* $8\frac{1}{2}$*oz.* avoirdupois: hence 2 butts, weighing 19*cwt.* 3*qrs.* 23*lb.* 1*oz.* (or nearly 20*cwt.*) are called a tun, or ton.

DRY MEASURE.

58 Dry Measure is used for all measurable commodities, except liquors [y].

2 pints	*make*	1 quart.
2 quarts	......	1 pottle. *pot.*
2 pottles	.....	1 gallon.
2 gallons	.....	1 peck. *pk.*
4 pecks	......	1 bushel. *bu.*
2 bushels	.....	1 strike. *str.*
4 bushels	.....	1 coom, or sack of corn. *s.*
2 cooms	......	1 quarter. *qr.*
5 quarters	......	1 wey or load.
2 weys	....	1 last.

pints.
8= 1 gallon.
16= 2= 1 peck.
64= 8= 4= 1 bushel.
256= 32= 16= 4= 1 coom.
512= 64= 32= 8= 2= 1 quarter.
2560=320=160=40=10= 5=1 wey.
5120=640=320=80=20=10=2=1 last.

[y] The standard weights and measures were formerly kept at Winchester, as they are now at the Exchequer. By a law of king Edgar, made nearly a century before the Conquest, it was enacted, that all measures of capacity should agree with the measures kept at Winchester. Hence the measure for dry goods is called *Winchester* measure. The gallon contains $268\frac{4}{5}$ cubic inches, and consequently the pint will be $33\frac{3}{5}$ cubic inches nearly.

The dry gallon is somewhat greater than a mean between the wine and beer gallons.

By an act of Parliament, made in 1697, it was ordered, that the standard bushel should be a cylinder of $18\frac{1}{2}$ inches diameter, and 8 inches deep, which are the legal dimensions of the corn bushel at present in use.

A bushel of wheat weighs 1000 ounces, or $62\frac{1}{2}$ pounds; a bushel of barley, 50 pounds; a bushel of oats, 38 pounds; a bushel of flour or salt, 56 pounds. A sack of flour is 5 bushels, and weighs 280 pounds.

Both 5 bushels, and 40 bushels, of corn are reckoned a load; the former for a man, the latter for a cart.

It may be remarked, that any measure *heaped up* is allowed to contain one third more than when it is *struck off*.

COALS[*].

4 pecks make 1 bushel.
3 bushels . . . . 1 sack. *sa.*
12 sacks . . . . 1 chaldron. *ch.*
21 chaldrons . . . . . . 1 score. *sc.*

pecks.
4= 1 bushel.
12= 3 = 1 sack.
144= 36 = 12 = 1 chaldron.
3024=756=252=21=1 score.

86. In 5 quarters 4 bushels 3 pecks of wheat, how many gallons?

OPERATION.

```
qr. bu. pk.
 5   4   3
 8
 ---
 44 bushels.
  4
 ---
179 pecks.
  2
 ---
Ans. 358 gallons.
```

Explanation.

I multiply first by 8, because (2 cooms, or) 8 bushels are a quarter, and take in the 4; next by 4, and take in 3; and lastly by 2, where there is nothing to take in.

87. In 4321 pecks of coals, how many chaldrons?

OPERATION.

```
pecks.
 4)4321(1
36)1080(30ch. 0bu. 1pk. Ans.
   108
   ---
     0
```

Explanation.

I first divide by 4 for bushels, and there is 1 peck over; then by 36, which gives 30 chaldrons, and nothing over.

88. In 15 quarters of wheat, how many quarts? *Ans.* 3840.

89. In 91 lasts of corn, how many pecks? *Ans.* 29120.

90. In 12*ch.* 3*bu.* 1*pk.* of coals, how many pecks? *Ans.* 1741.

91. In 537 pecks of wheat, how many quarters? *Ans.* 16*qr.* 6*bu.* 1*pk.*

92. How many pecks are there in 20 score of coals? *Ans.* 60480.

---

[*] The standard coal bushel is 8 inches deep, and $19\frac{1}{2}$ wide: the measure is always heaped up; and for every 5 chaldrons bought at one time the seller must give 63 sacks.

59. GOODS SOLD BY TALE.

2 things	*are*	a pair, couple, or brace.
12 things		a dozen.
13 things		a long dozen.
20 things		a score.
12 dozen		a groce.
12 groce		a great groce.
5 score		a hundred.
6 score		a great hundred.
12 hundred of some things		a thousand.

60. PAPER, PARCHMENT, AND BOOKS.

24 sheets of paper	*make*	a quire.
20 quires		a ream.
$21\frac{1}{2}$ quires		a printer's ream.
2 reams		a bundle.
5 dozen skins of parchment		a roll.
72 words in common law		a sheet.
80 —— in the Exchequer		a sheet.
90 —— in Chancery		a sheet.

Of the different sizes of books, folio is the largest.

A sheet of paper makes in

folio (fol.)	2 leaves, or 4 pages.
quarto (4to.)	4 leaves, or 8 pages.
octavo (8vo.)	8 leaves, or 16 pages.
duodecimo (12mo.)	12 leaves, or 24 pages.
octodecimo (18mo.)	18 leaves, or 36 pages.
twenty-fours (24to.)	24 leaves, or 48 pages.

93. In 50 dozen of eggs, how many score? *Ans.* 30.

94. In 720 score of corks, how many groce? *Ans.* 100.

95. How many great groce are there in one hundred thousand? *Ans.* 57 *g. groce,* 1504 *over.*

96. In forty great hundred of tallies, how many dozen? *Ans.* 400.

97. How many score in one hundred long dozens? *Ans.* 65.

98. In twenty thousand lemons, how many long dozens? *Ans.* 1538 *l. doz.* 6 *over.*

[a] Oranges, lemons, lead-pencils, tobacco-pipes, corks, &c. are usually sold by the groce; eggs, tallies, nails, and many other small articles, are sold by the great hundred.

99. How many sheets of paper are there in a folio volume of 360 pages? *Ans.* 90.

100. How many sheets of paper are there in one thousand reams? *Ans.* 480000.

101. A quarto work contains 12600 pages; how many sheets of paper are there in it? *Ans.* 1575.

102. Thirty-five sheets are employed in an octavo book; how many pages does it contain? *Ans.* 560.

103. How many common reams are there in fifty printer's reams of paper? *Ans.* 53 *reams,* 15 *quires.*

104. An octavo work of 20 sheets contains eight hundred thousand letters; how many are there in a page? *Ans.* 2500.

61. TIME[h].

60 seconds (″)	*make*	1 minute. *m.*
60 minutes	....	1 hour. *h.*
24 hours	....	1 day. *d.*
7 days	....	1 week. *w.*
4 weeks	....	1 month. *mo.*
365 days 6 hours	....	1 Julian year.
13*mo.* 1*d.* 6*h.* or 52*w.* 1*d.* 6*h.*		1 Julian year.

[h] A day is the length of time which elapses while the earth revolves once about its axis, which is about 23*h.* 56*m.* 4″. although commonly reckoned 24 hours. Hours, minutes, and seconds, are divisions and subdivisions of a day, an hour being one twenty-fourth part of a day; a minute, one sixtieth part of an hour; and a second, one sixtieth part of a minute. The week is a religious institution, appointed by the ALMIGHTY immediately after the creation; and the observance of every seventh day as a day of holy rest is repeatedly enjoined in the Scriptures.

A month is properly a portion of time regulated by the moon: thus a lunar periodical month is 27*d.* 7*h.* 43*m.* 8″. being the time the moon takes in going from any point in the ecliptic to the same point again.

A *lunar synodical month* or *lunation* consists of 29*d.* 12*h.* 44*m.* 3″. 11‴. being the space of time which passes from one new moon to the next; to these may be added the *solar month,* of 30*d.* 10*h.* 29*m.* 5″. which is the time the sun takes to pass through one sign, or one twelfth part of the ecliptic; and likewise the *civil month* of 28 days, as in the Table. A year is the space of time in which the earth makes one complete revolution round the sun, and in which all the seasons return: this is called the *solar year,* and consists of 365*d.* 5*h.* 48*m.* 48″. The Julian year of 365*d.* 6*h.* is commonly reckoned 365 days only; and for

seconds					
60=	1 minute.				
3600=	60	= 1 hour.			
86400=	1440	= 24	= 1 day.		
604800=	10080=	168=	7	=1 week.	
2419200=	40320=	672=	28=	4=	1 month.

105. In 12 weeks, how many seconds? *Ans.* 7257600.
Multiply by 7, 24, 60, *and* 60.

106. In 1234567 seconds, how many weeks? *Ans.* 2*w.* 6*h.* 56*m.* 7″. *Divide by* 60, 60, 24, *and* 7.

107. In 1*mo.* 2*w.* 3*d.* 4*h.* 5*m.* how many minutes? *Ans.* 65045.

108. How many seconds are there in a Julian year? *Ans.* 31557600.

109. October the 25th, 1809, the King completed the 49th year of his reign; how many minutes are there in that space of time, reckoning Julian years? *Ans.* 25772040.

110. How many hours have elapsed since the birth of Christ to Christmas 1810, allowing the years to be of the Julian kind?

COMPOUND ADDITION.

62. A compound number is that which consists of different denominations in money, weights, measures, &c.

---

the odd 6 hours, a day is added to February every fourth year, which year is called *Bissextile*, or *Leap year*. Thus February, in the Leap year, has 29 days; and consequently the Leap year consists of 366.

To find Leap year, this is the rule.

Divide the date by *four*, and you'll discover
That 'tis *Leap year*, if *nought* remains at last;
But *one*, or *two*, or *three*, remaining over
Denote that *just so many years 'tis past*.

To find the number of days in each month.

Thirty days hath September,
April, June, and November;
February has twenty-eight alone,
The other months have thirty-one:
But Leap year comes one year in four,
And Febr'ary then has one day more.

Compound Addition teaches to find the sums of such compound numbers as are of the same kind.

RULE I. Place the numbers to be added so that all those of the same denomination may stand under one another in a column; and let two dots (thus . .) be put between each two numbers of different denominations.

II. Add all the numbers in the least denomination together, and reduce the sum to the next higher denomination, and set down the remainder, if any.

III. Carry the number arising from this reduction to the next superior denomination; add it up, reduce the sum to the next superior denomination, set down the remainder, carry, &c. as before.

IV. Proceed in this manner with all the denominations to the highest, which must be added and put down like simple Addition.

Method of Proof. Cut off the top line, and proceed as in simple Addition[c].

---

[c] Here we add up each separate denomination by simple Addition, and the truth of the rule may be shewn from any of the examples included under it. We will take the first example in Money, in which the sum of the farthings is 11; the sum of the pence 26; the sum of the shillings 71; and the sum of the pounds 224. Now as we always estimate any sum, namely, pounds, shillings, pence, or farthings, in the highest of these denominations it is reducible to, these farthings, pence, and shillings, must if possible be reduced higher. Let us try.

Thus, 11 farthings	=	0	0	$2\frac{3}{4}$	by Art. 46.
26 pence	=	0	2	2	
71 shillings	=	3	11	0	
224 pounds	=	224	0	0	
These added give		227	13	$4\frac{3}{4}$	as in the example.

Here note, that the 2 pence in the first line above is the 2 carried from farthings to pence, (in Ex. 1.) the 2 shillings in the second line is the 2 carried from pence to shillings; and the 3 pounds in the third line is the 3 carried from shillings to pounds: if this illustration be well understood, the reason of the following modes of operation will be extremely plain.

63. MONEY[d].

Farthings, Pence, and Shillings, Tables.

FARTHINGS.		PENCE.						SHILLINGS.			
q.	*d.*	*d.*	*s.*	*d.*	*d.*	*s.*	*d.*	*s.*	*L.*	*s.*	*d.*
4 *are*	1	12 *are*	1	0	80 *are*	6	8	20 *are*	1	0	0
6	$1\frac{1}{2}$	20	1	8	84	7	0	30	1	10	0
8	2	24	2	0	90	7	6	40	2	0	0
10	$2\frac{1}{2}$	30	2	6	96	8	0	50	2	10	0
12	3	36	3	0	100	8	4	60	3	0	0
14	$3\frac{1}{2}$	40	3	4	108	9	0	70	3	10	0
16	4	48	4	0	110	9	2	80	4	0	0
18	$4\frac{1}{2}$	50	4	2	120	10	0	90	4	10	0
20	5	60	5	0	130	10	10	100	5	0	0
22	$5\frac{1}{2}$	70	5	10	132	11	0	110	5	10	0
24	6	72	6	0	144	12	0	120	6	0	0

[d] The coins used in England are gold, silver, and copper; the gold coins are, a guinea, halfguinea, and seven shilling piece. The silver coins are, a crown, halfcrown, shilling, and sixpence. The copper coins are, a twopenny piece, penny, halfpenny, and farthing. These are called *real* coins; but any denomination of money, which is not represented by a single coin, is called *imaginary;* thus a guinea, a crown, &c. are *real* coins; but a pound, a groat, &c. (having no single piece that will represent them) are *imaginary.*

The moneyers suppose any quantity of gold divisible into 24 equal parts, which they call *carats,* and each carat they divide into 24 parts, calling these *grains of a carat;* by this they denominate the fineness of their gold. If the gold be free from any mixture (called alloy), it is said to be 24 carats fine; but if there be 2 carats (out of the 24) of alloy in it, the gold is said to be 22 carats fine, &c.

The standard for British gold coin is 22 carats, namely, 22 of pure gold, and 2 of alloy, composed of silver and copper.

The standard for silver coin is 11oz. 2dwts. of pure silver, mixed with 18dwts. of copper alloy.

The value and weight of Coin.

		L.	*s.*	*d.*		*dwt.*	*gr.*
GOLD.—A Guinea	*value*	1	1	0	*weighs*	5	$9\frac{1}{2}$
Halfguinea		0	10	6		2	$16\frac{3}{4}$
Seven Shilling Piece						1	19
SILVER.—A Crown		0	5	0		19	$8\frac{1}{2}$
Halfcrown		0	2	6		9	$16\frac{1}{4}$
Shilling						3	21
Sixpence						1	$22\frac{1}{2}$

A pound of standard gold makes $44\frac{1}{2}$ guineas, and a pound of standard silver 62 shillings.

EXAMPLES.

1.

	L.	s.	d.
	35	12	8¾
	21	17	6½
	52	14	3¼
	47	13	5¾
	69	15	4½
Sum	227	13	4¾
	192	0	8
Proof	227	13	4¾

Explanation.

I begin at the bottom of the farthings, and say 2 and 3 are 5 and 1 are 6 and 2 are 8 and 3 are 11; 11 farthings reduced to pence are 2 pence three farthings; I put down ¾ and carry 2 to the pence, which added up amount to 28; this reduced to shillings is 2*s.* 4*d.* I put down 4, and carry 2 to the units line of shillings; the shillings being added (the units first, and then the tens) give 73 shillings; these reduced to pounds are 3*l.* 13*s.* put down 13 and carry 3 to the pounds, which are added up exactly like an example in simple Addition. The first line of the work being finished, I cut off the top line (namely, 35*l.* 12*s.* 8*d.*¾) by drawing a light stroke under it. I then add up the same numbers again which I added before, all except the top line cut off, in the same manner as before, and the result is the second line of the work; I lastly add the said second line and the top line cut off together, and the sum will be the third line of the work; this third line agreeing in every particular with the first line, shews that the operation is right.

2.

	L.	s.	d.
	27	14	8½
	38	15	9¼
	41	17	7½
	52	18	2¼
	27	10	3¾
Sum	188	16	7¼
	161	1	10¾
Proof	188	16	7¼

3.

L.	s.	d.
35	2	3¼
51	4	6¼
13	1	1½
27	2	2¼
32	1	1¼
158	11	2½
123	8	11¼
158	11	2½

4.

L.	s.	d.
24	11	2¾
41	11	11½
27	17	10¼
71	19	11¾
98	18	10¾
264	19	11
240	8	8¼
264	19	11

A pennypiece, 2 new halfpence, or 3 old ones, should weigh an ounce Avoirdupois.

When the Saxons first settled in Britain, they were called *Easterlings*, (from the circumstance of their coming from the east,) and their money, *easterling* money: hence by a corruption frequent in language the word *sterling* is derived. But it is remarkable, that a term, which at that time was applied exclusively to express something *foreign*, should by the lapse of time completely change its signification, so as to denote exclusively that which is *British*.

5.

L.	s.	d.
39	1	2¼
24	2	7½
41	9	3¼
12	8	4¼
21	4	1¼

6.

L.	s.	d.
48	10	7½
21	11	1½
12	12	2¼
21	13	2¼
21	15	6¼

7.

L.	s.	d.
56	7	8
64	2	6½
41	3	1¼
92	2	4
31	9	7¾

8.

L.	s.	d.
91	8	10¼
14	9	6
72	8	9¾
21	9	8
86	9	7¾

9.

L.	s.	d.
73	12	6¾
10	10	4½
68	16	9¾
84	11	6½
47	10	3¼

10.

L.	s.	d.
14	10	0½
23	11	9¼
39	12	1¼
17	4	0¼
42	0	3¼

11.

L.	s.	d.
83	13	11¼
37	16	10½
47	10	10¼
74	11	11¼
12	10	10¼

12.

L.	s.	d.
72	0	0¾
49	6	7
98	9	0¾
87	9	0¾
74	8	1¾

13.

L.	s.	d.
14	5	6½
76	12	10¾
64	1	11¾
47	16	8¾
58	18	1¾

14.

L.	s.	d.
20	19	11¾
47	0	0¾
4	14	8¾
0	19	0¾
0	0	11¾

15.

L.	s.	d.
123	18	10½
49	1	11¾
713	14	10¾
427	12	11¾
9	19	1½

16.

L.	s.	d.
374	17	9¼
9	9	9¾
10	10	10¾
28	19	11½
475	12	8½

64. TROY WEIGHT.

17.

lb.	oz.	dwt.	gr.
21	11	12	14
32	10	11	12
51	11	13	15
64	10	15	12
13	11	11	10
185	8	4	15
163	8	12	1
185	8	4	15

18.

lb.	oz.	dwt.	gr.
4	3	8	9
2	4	9	15
3	9	7	8
5	8	9	7
8	9	18	11
24	11	13	2
20	8	4	17
24	11	13	2

Explanation of Ex. 17.

I first add up the grains, and find they amount to 63; this I divide by 24, which goes twice in it, and 15 over; I put down 15, and carry 2 to the dwts.; these added up amount to 64; this I divide by 20, exactly as was done in shillings, and find it goes 3 times, with 4 over; put down 4, and carry 3 to the ounces, which added up amount to 56; this is divided by 12, as was done in pence, and the quotient is 4, with 8 over; put down 8, and carry 4 to the pounds, which are added and put down like simple addition. Thus the first line is found. In the second line, the grains amount to 49, which is twice 24, and 1 over; put down 1, and carry 2. The dwts. amount to 52, which is twice 20, and 12 over; put down 12, and carry 2. The ounces amount to 44, which is 3 times 12, and 8 over; put down 8, and carry 3 to the pounds, which are added by simple addition. The proof is done exactly as in Money, except that the numbers for which you carry differ.

19.

lb.	oz.	dwt.	gr.
73	1	2	3
49	7	4	8
12	9	8	7
53	8	9	4
67	2	3	9

20.

lb.	oz.	dwt.	gr.
4	9	8	7
1	8	7	9
3	4	17	8
9	11	8	10
7	10	12	20

21.

lb.	oz.	dwt.	gr.
12	4	17	23
16	11	10	12
78	10	11	13
23	9	8	7
45	8	3	4

65. APOTHECARIES' WEIGHT.

22.

℔.	℥	ʒ	℈	gr.
1	2	1	2	10
3	7	2	1	15
6	9	6	2	12
9	7	4	1	10
2	4	6	1	10
23	7	6	0	17
22	5	4	1	7
23	7	6	0	17

23.

℔.	℥	ʒ	℈
4	2	8	1
7	5	6	2
8	9	7	1
6	4	3	2
9	11	0	0
36	9	5	0
32	7	1	2
36	9	5	0

Explanation of Ex. 22.

The grains amount to 57, or 2 scruples 17 grains. I put down 17, and carry 2 to the scruples; these amount to 9, that is, to 3 drams; I put down 0, and carry 3 to the drams; these amount to 22, or 2 ounces, and 6 drams over; put down 6, and carry 2 to the ounces; the ounces added amount to 31, or 2 pounds, and 7 ounces over; put down 7, and carry 2 to the pounds, which are added as before. The second line, and proof, from what has been said, will be easily understood.

24.

℥	ʒ	℈	gr.
71	1	1	3
25	3	0	4
78	6	2	6
41	4	1	9
32	7	2	8

25.

℔	℥	ʒ	℈
14	10	7	2
25	9	6	1
79	11	4	0
4	3	2	1
30	1	0	0

26.

℔	℥	ʒ	℈	gr.
8	9	5	2	13
7	8	1	2	10
9	11	7	1	2
4	0	7	2	9
6	7	4	0	12

66. AVOIRDUPOIS WEIGHT.

27.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
7	6	3	4	5	6
4	9	2	8	9	9
5	9	1	9	8	7
8	4	2	9	8	9
2	5	1	1	2	3
27	15	2	5	2	2
20	8	3	0	12	12
27	15	2	5	2	2

28.

cwt.	*qr.*	*lb.*	*oz.*
4	1	13	12
6	0	27	9
8	1	26	8
7	2	24	9
3	3	21	9
30	3	1	15
26	1	16	3
30	3	1	15

Explanation of Ex. 27.

The drams added amount to 34, or 2 ounces, and 2 drams over; put down 2, and carry 2. The ounces likewise amount to 34, or 2lb. 2oz.; put down 2, and carry 2. The pounds amount to 33, or 1qr. 3lb.; put down 5, and carry 1. The quarters amount to 10, or 2cwt. 2qr.; put down 2, and carry 2. The cwts. amount to 35, or 1t. 15cwt.; put down 15, and carry 1 to the tons, which amount to 27. There will be no difficulty in the second line and proof.

29.

t.	*cwt.*	*qr.*	*lb.*
1	2	3	4
4	3	2	1
8	12	1	20
4	10	2	20
2	11	1	2

30.

cwt.	*qr.*	*lb.*	*oz.*
11	1	12	10
10	2	10	10
21	3	11	12
10	1	2	1
8	0	0	4

31.

lb.	*oz.*	*dr.*
21	15	12
17	14	13
40	10	15
32	11	14
28	10	10

67. LONG MEASURE.

32.

lea.	*m.*	*fur.*	*p.*	*yd.*	*f.*	*in.*	*bc.*
47	1	2	3	2	1	7	1
14	2	3	9	1	2	3	2
21	1	5	30	1	2	4	1
15	2	3	4	2	1	3	1
27	2	4	9	4	1	2	3
127	1	2	17	1	2	9	2
80	0	0	13	$4\frac{1}{2}$	1	2	1
127	1	2	17	1	2	9	2

Explanation.

The yards in the first line amount to 12, which contains $5\frac{1}{2}$ twice, and 1 over; I put down 1, and carry 2. In the second line the yards amount to 10, that is, *once* $5\frac{1}{2}$, and $4\frac{1}{2}$, over; put down $4\frac{1}{2}$, and carry 1: the rest is obvious.

33.

lea.	m.	fur.	p.
10	1	2	34
12	1	2	12
41	2	1	10
27	1	2	11
14	2	5	9

34.

yd.	f.	in.	bc.
28	1	2	1
17	1	3	2
31	2	3	2
76	2	4	1
20	2	7	1

35.

yd.	f.	in.	l.
60	1	2	11
71	1	1	10
17	1	6	10
14	2	9	11
27	1	8	10

68. CLOTH MEASURE.

36.

yd.	qr.	n.
28	3	3
71	2	1
16	3	3
42	1	2
23	2	3
183	2	0

37.

yd.	qr.	n.
42	1	2
28	0	0
35	3	3
60	2	3
78	3	3

38.

FE.	qr.	n.
83	2	0
12	1	2
24	2	3
53	2	3
37	2	2

39.

EE.	qr.	n.
39	4	3
13	4	2
48	3	3
92	1	0
50	2	1

40.

Fr.E.	qr.	n.
98	5	3
16	4	3
79	3	3
90	4	2
25	3	3

69. SQUARE MEASURE.

41.

a.	r.	p.	yd.	f.	in.
4	3	12	20	1	40
7	3	20	5	8	16
8	1	10	6	4	80
9	2	12	20	2	60
5	1	10	16	4	10
35	3	26	8½	2	62

42.

a.	r.	p.
12	1	10
76	2	15
14	1	28
16	3	4
20	0	19

43.

a.	r.	p.
131	1	14
127	2	17
300	0	0
123	1	16
208	3	39

44.

yd.	f.	in.
710	8	100
100	2	100
276	7	47
701	8	24
101	8	123

In Ex. 41. the yards amount to 69; there are twice 30¼ in this number, and 8½ over; put down 8½, and carry 2: all the rest is obvious.

70. CUBIC MEASURE.

ROUGH TIMBER.

45.

ld.	f.	in.
123	26	727
212	17	497
376	12	741
471	14	800
124	20	764
1309	10	73

46.

ld.	f.	in.
27	10	100
14	12	914
76	16	416
64	17	804
12	12	123

Explanation of Ex. 45.

The inches amount to 3529, which divided by 1728, the quotient is 2, and 73 over; put down 73, and carry 2 to the yards; these being added, the sum is 91; this divided by 27, the quotient is 3, and 10 over; put down 10, and carry 3.

HEWN TIMBER.

47.

ld.	f.	in.
48	12	1000
10	10	1000
14	15	1600
15	20	1200
31	13	1000

48.

ld.	f.	in.
32	20	1600
13	13	1300
30	21	1000
22	10	400
78	10	100

71. WINE MEASURE.

49.

t.	hhd.	gal.	qt.
2	3	12	3
1	1	21	1
7	2	10	2
9	1	10	2
4	3	20	2
25	3	12	2

50.

gal.	qt.	pt.
13	2	1
16	1	1
47	2	1
10	1	1
71	2	1

51.

tier.	gal.	qt.
42	10	1
10	20	2
13	10	3
11	11	1
14	20	1

52.

qt.	pt.	gills
14	1	2
12	0	1
70	1	1
14	1	3
10	1	1

53.

a.	gal.	qt.
61	1	1
17	9	2
23	1	1
15	2	2
60	1	1

72. BEER MEASURE.

54.

butts	hhd.	bar.	kild.
8	1	1	1
9	1	1	1
4	1	0	1
7	0	0	1
6	1	1	0
37	0	2	0

55.

fir.	gal.	qt.
17	1	2
60	4	3
45	3	1
11	1	2
76	8	3

56.

gal.	qt.	pt.
48	1	1
18	3	0
40	2	0
76	3	1
16	1	1

57.

hhd.	gal.	qt.
53	14	1
12	12	3
76	10	1
31	11	2
24	18	2

58.

bar.	gal.	qt.
37	1	1
76	5	2
47	28	2
12	12	1
27	16	1

73. DRY MEASURE.

CORN.

59.

qr.	bu.	pk.	gal.
32	7	3	1
17	4	3	1
74	2	2	1
41	5	1	1
26	6	2	1
193	3	1	1

60.

lasts	wey	qr.	bu.	p.
25	1	4	4	3
18	1	1	1	1
24	1	3	7	1
56	1	4	3	1
14	1	3	2	1

COALS.

61.

ch.	bu.	pk.
29	20	3
12	12	2
31	10	3
48	20	2
13	16	1

62.

sc.	ch.	sa.	bu.
4	12	3	2
7	10	1	2
8	10	9	1
9	20	11	2
5	20	11	1

74. TIME.

63.

w.	d.	h.	m.	"
3	1	12	18	20
1	4	20	20	10
2	5	10	10	20
4	6	20	10	20
7	4	12	40	50
20	2	3	40	0

64.

mo.	w.	d.	h.
8	1	1	11
9	3	6	19
1	3	4	27
2	1	2	12
3	3	1	20

65.

mo.	w.	d.	h.
11	1	3	20
14	3	6	21
16	1	4	22
84	2	6	20
73	3	5	20

66.

d.	h.	m.	"
27	10	12	18
31	12	20	30
47	20	30	40
71	10	41	52
14	15	16	17

PROMISCUOUS EXAMPLES FOR PRACTICE.

67.

L.	s.	d.
123	12	$10\frac{3}{4}$
217	13	$11\frac{1}{2}$
718	10	$10\frac{1}{2}$
124	11	$11\frac{1}{2}$
273	16	$8\frac{3}{4}$
731	17	$9\frac{3}{4}$
314	19	$8\frac{3}{4}$
471	16	$9\frac{1}{2}$
784	12	$7\frac{3}{4}$
149	16	$7\frac{1}{2}$

68.

L.	s.	d.
748	18	$11\frac{1}{2}$
479	19	$11\frac{3}{4}$
90	19	$10\frac{3}{4}$
417	18	$11\frac{1}{2}$
9	17	$9\frac{3}{4}$
704	18	$9\frac{3}{4}$
27	16	$10\frac{3}{4}$
8	19	$8\frac{3}{4}$
39	18	$11\frac{1}{2}$
495	14	$11\frac{3}{4}$

69.

lb.	oz.	dwt.	gr.
27	11	19	23
4	10	19	22
17	11	14	23
44	11	18	21
9	10	11	12
33	11	12	13
9	10	11	12
47	11	18	20
13	10	17	21
37	11	12	15

70.

t.	cwt.	qr.	lb.	oz.	dr.
47	19	3	27	15	14
4	13	3	26	14	10
9	16	2	20	15	15
87	12	3	27	10	15
76	10	1	21	12	14
7	11	1	24	14	15
64	10	3	20	12	13
9	19	2	10	11	15
7	18	2	11	13	15
24	18	3	10	10	14

71.

℔	℥	ʒ	℈	gr.
17	11	7	2	3
41	10	7	2	2
12	11	6	2	14
37	10	6	2	17
12	11	7	2	16
34	10	5	1	19
14	10	4	2	18
24	11	7	2	17
56	10	6	1	15
19	11	7	2	19

72.

E.E.	qr.	n.
102	4	3
97	4	3
2	3	3
8	4	3
12	1	2
39	4	3
176	3	2
475	3	3
88	4	3
7	2	2

73.					74.			
lea.	*m.*	*fur.*	*p.*		*yd.*	*f.*	*in.*	*b.*
47	2	7	37		374	2	11	11
12	1	6	20		746	2	10	11
76	2	7	31		90	1	2	10
9	2	6	27		474	2	10	11
8	2	7	39		8	1	9	10
764	2	7	38		906	2	10	11
28	2	6	32		39	2	11	11
497	2	7	28		484	2	10	11
9	1	5	30		98	2	10	11
46	2	7	35		9	2	8	10

75. I am indebted to A, 10*l.* 11*s.* 6*d.*; to B, 29*l.* 19*s.* 0*d.*; to C, 129*l.* 0*s.* 11*d.*$\frac{1}{4}$; to D, 3795*l.* 18*s.* 10*d.*$\frac{1}{4}$; to E, 789*l.* 19*s.* 9*d.*$\frac{1}{2}$; and to F, 239*l.* 19*s.* 8*d.*$\frac{1}{4}$; what sum do I owe in all. *Ans.* 4995*l.* 9*s.* 10*d.*$\frac{1}{4}$.

76. A poor widow received in charitable donations as follows; viz. from F, ten guineas; from G, three moidores: from H, one hundred pounds; from K, fifty shillings; from T, eleven crown pieces; from X, seventeen guinea notes; and from Z, a check on his banker for threescore and ten pounds; what did she receive in all. *Ans.* 207*l.* 13*s.*

77. What does my house stand me in per year, supposing the rent is forty pounds ten shillings; the assessed taxes, fifteen guineas and sixpence; land tax, three pounds five shillings; poor's rate, six pounds seven and fourpence; and church rate, highway rate, &c. two guineas and a half? *Ans.* 68*l.* 10*s.* 4*d.*

78 A silversmith sold plate as follows; on Monday he sold 7*lb.* 11*oz.* 19*dwts.* 17*gr.*; on Tuesday, 12*lb.* 10*oz.* 17*dwts.* 23*gr.*; on Wednesday, 9*lb.* 11*oz.* 12*dwts.* 20*gr.*; on Thursday, 15*lb.* 11*oz.* 14*dwts.* 21*gr.*; on Friday, 23*lb.* 9*oz.* 17*dwts.* 15*gr.*; and on Saturday, 11*lb.* 10*oz.* 13*dwts.*; what quantity did he sell during the whole week? *Ans.* 82*lb.* 6*oz.* 16*dwts.*

79. A wholesale grocer sent the following quantities of sugar to five customers; viz. to A, 3*cwt.* 3*qr.* 27*lb.*; to B, 5*cwt.* 2*qr.* 26*lb.*; to C, 1*cwt.* 2*qr.* 2*lb.*; to D, 4*cwt.* 3*qr.* 25*lb.*; and to E, 3*qr.* 26*lb.*; what quantity did he send in all? *Ans.* 17*cwt.* 22*lb.*

80. A person bought six pieces of Irish linen; the first containing 7*yds.* 3*qr.*; the second, 19*yds.* 3*qr.* 3*n.*; the third,

28*yds*. 3*qr*. 2*n*.; the fourth, 33*yds*. 2*qr*.; the fifth, 6*yds*. 2*qr*. 3*n*.; and the sixth, 10*yds*. 3*qr*. 3*n*.; what was the whole quantity bought? *Ans*. 107*yds*. 2*qr*. 3*n*.

81. A common was ordered to be inclosed, and it was accordingly divided into 6 meadows; the first containing 27*a*. 3*r*. 31*p*.; the second, 14*a*. 3*r*.; the third, 32*a*. 2*r*. 25*p*.; the fourth, 15*a*. 3*r*. 18*p*.; the fifth, 32*a*. 1*r*. 19*p*.; and the sixth, 3*r*. 39*p*.; there were also 12*a*. 2*r*. 11*p*. allotted for a road through the ground; what quantity of land did the common contain? *Ans*. 137*a*. 23*p*.

82. A farmer bought several quantities of wheat as follows; of A he bought 3*ld*. 2*qr*. 5*bu*.; of B, 3*qr*. 7*bu*. 3*pk*.; of C, 11*ld*. 1*qr*.; of D, 7*bu*. 3*pk*.; of E, 4*qr*. 6*bu*. 3*pk*.; and of F, 14*ld*. 4*qr*. 3*pk*.; how much did he buy in all? *Ans*. 31*ld*. 2*qr*. 4*bu*.

83. A coal-merchant received five lighters loaded with coals from the pool; the first contained 12*ch*. 3*bu*. 3*pk*.; the second, 19*ch*. 27*bu*. 3*pk*.; the third, 17*ch*. 35*bu*. 2*pk*.; the fourth 11*ch*. 29*bu*. 3*pk*.; and the fifth, 23*ch*. 32*bu*. 3*pk*.; how many chaldrons did he receive? *Ans*. 85*ch*. 21*bu*. 2*pk*.

COMPOUND SUBTRACTION.

76. Compound Subtraction teaches to find the difference of two given compound numbers of the same kind, by taking the less from the greater.

RULE. Place the less compound number below the greater, so that like denominations may stand under each other as in Compound Addition.

Beginning at the least denomination, subtract the lower numbers from those above, putting each remainder under its respective denomination.

When the lower number of any denomination is greater than the upper, increase the upper by as many as make one of the next superior denomination, subtract the lower number from this sum, set down the remainder, and carry 1 to the next lower number; subtract, and proceed in this manner until the work is finished*.

* When the inferior denominations in the upper line are respectively greater than those in the lower, the reason of the rule is plain; when they are less, the borrowing and carrying depend on the same principles as common subtrac-

Method of Proof.

Add the remainder (or number arising from the operation) and the lesser number together, and if the sum be like the greater, the work is right.

EXAMPLES.

1. From 123*l.* 4*s.* 5*d.*$\frac{1}{4}$ take 102*l.* 19*s.* 6*d*$\frac{3}{4}$.

OPERATION.

	L.	s.	d.
	123	4	$5\frac{1}{4}$
	102	19	$6\frac{3}{4}$
Rem.	20	4	$10\frac{1}{2}$
Proof	123	4	$5\frac{1}{4}$

Explanation.

Here I begin at the farthings, and say, 3 from 1 I cannot, I therefore borrow 4 farthings, which added to the 1 make 5, then 3 from 5, and 2 remain; put down $\frac{1}{2}$, and carry 1 to the 6 makes 7; then I say, 7 from 5 I cannot, borrow 12 to the 5 which make 17, then 7 from 17, and 10 remain; put down 10, and carry 1 to 19 make 20; then 20 from 4 I cannot, therefore I borrow 20 to the 4, making 24, then 20 from 24, and 4 remain; put down 4, and carry 1 to the 2: the rest is merely simple subtraction. The proof arises from adding the remainder and the line next above it together by Compound Addition.

	2. L.	s.	d.	3. L.	s.	d.	4. L.	s.	d.
From	471	7	$8\frac{1}{2}$	76	1	$11\frac{3}{4}$	47	18	$1\frac{1}{2}$
Take	101	1	$2\frac{1}{4}$	14	9	$1\frac{1}{2}$	10	19	$9\frac{3}{4}$
Rem.	370	6	$6\frac{1}{4}$	61	12	$10\frac{1}{4}$	36	18	$3\frac{3}{4}$
Proof	471	7	$8\frac{1}{2}$	76	1	$11\frac{3}{4}$	47	18	$1\frac{1}{2}$

5. L.	s.	d.	6. L.	s.	d.	7. L.	s.	d.
12	11	$6\frac{3}{4}$	78	9	$4\frac{1}{2}$	40	19	$3\frac{1}{4}$
10	1	$2\frac{1}{4}$	16	1	$2\frac{1}{2}$	18	3	$7\frac{1}{4}$

---

tion does; so in Ex. 1. we borrow 4 (or 1 penny) in farthings, to compensate which we carry 1 (penny) to the pence; we borrow 12 (or 1 shilling) in pence, to compensate which we carry 1 (shilling) to the shillings; and in shillings we borrow 20, (or 1 pound,) carrying 1 afterwards to the pounds. That these carryings always exactly compensate for the number borrowed appears plain, for carrying 1 to the lower number is in effect the same as taking from the upper the 1 borrowed. The same reasoning may be applied to every kind of examples that can be proposed under this rule.

8.			9.			10.		
L.	s.	d.	L.	s.	d.	L.	s.	d.
73	6	8¼	62	15	5¼	84	3	2¼
45	11	2¼	12	10	9½	76	8	7½

11.			12.			13.		
L.	s.	d.	L.	s.	d.	L.	s.	d.
59	6	10¼	39	17	4½	24	0	11
8	18	2¾	10	0	6	15	3	3¼

14.			15.			16.		
L.	s.	d.	L.	s.	d.	L.	s.	d.
43	14	1	48	0	0½	67	1	2¾
22	11	10¾	47	19	11½	1	2	3¼

77. TROY WEIGHT.

17.

	lb.	oz.	dwt.	gr.
From	97	1	2	3
Take	12	8	17	21
Rem.	84	4	4	6
Proof	97	1	2	3

Explanation.

I borrow 24 in grains, which added to 3 make 27, then 21 from 27, and 6 remain to be put down; carry 1 to 17 are 18, then 18 from 2 I cannot; borrow 20 to the 2 make 22, then 18 from 22, and 4 remain to put down; carry 1 to 8 are 9, then 9 from 1 I cannot; borrow 12 to the 1 make 13, then 9 from 13, and 4 remain to put down; carry 1, and proceed as in simple subtraction.

18.				19.				20.			
lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.
71	8	16	7	41	7	12	11	9	1	3	6
14	11	10	9	18	11	2	5	1	2	0	8
56	9	5	22								

20.				21.			
lb.	oz.	dwts.	gr.	lb.	oz.	dwts.	gr.
8	11	9	8	6	0	2	10
8	4	10	2	4	1	0	20

78. APOTHECARIES' WEIGHT.

23.

	℔.	℥	ʒ	℈	gr.
From	61	1	2	2	1
Take	20	4	7	2	10
Rem.	40	8	2	2	11
Proof	61	1	2	2	1

Explanation.

I have to borrow in every place throughout this example except in the pounds. I borrow 20 in grains, 3 in scruples, 8 in drams, and 12 in ounces; observing always to carry 1, after I have borrowed, to the next lower figure.

24.

℔.	℥	ʒ	℈	gr.
9	0	7	0	12
4	3	1	2	0
4	9	5	1	12

25.

℔.	℥	ʒ	℈	gr.
8	11	0	2	17
1	2	3	1	4

26.

℔.	℥	ʒ	℈
6	4	7	0
5	10	0	2

57.

℥	ʒ	℈	gr.
10	0	1	11
6	1	1	8

58.

℔.	℥	ʒ	℈	gr.
23	1	0	0	11
10	0	0	1	17

79. AVOIRDUPOIS WEIGHT.

29.

	t.	cwt.	qr.	lb.	oz.	dr.
From	90	1	2	3	4	5
Take	12	4	3	5	6	7
Rem.	77	16	2	25	13	14
Proof	90	1	2	3	4	5

Explanation.

I have to borrow in every place except the last; viz. I borrow 16 in drams, 16 in ounces, 28 in pounds, 4 in quarters, and 20 in cwts. observing always to carry 1 after borrowing.

30.

t.	cwt.	qr.	lb.
7	18	2	20
1	3	3	11
6	14	3	9

31.

cwt.	qr.	lb.	oz.
9	3	12	1
4	1	8	4

32.

qr.	lb.	oz.	dr.
3	13	14	2
1	2	13	7

33.

cwt.	qr.	lb.
18	1	2
13	3	27

34.

t.	cwt.	qr.	lb.	oz.	dr.
8	0	1	0	12	4
0	1	1	0	11	12

80. LONG MEASURE.

35.

lea.	*m.*	*fur.*	*p.*
27	2	1	8
20	2	5	23
6	2	3	25

36.

yd.	*f.*	*in.*	*bc.*
9	1	11	2
1	2	3	1

37.

fath.	*f.*	*in.*	*bc.*
12	1	2	0
3	0	7	2

In Ex. 35. I borrow 40 in poles, 8 in furlongs, and 3 in miles.

38.

cub.	*in.*	*l.*
471	1	2
36	16	8

39.

lea.	*m.*	*fur.*	*p.*
45	1	2	30
17	2	1	31

81. CLOTH MEASURE.

40.

yd.	*qr.*	*n.*
337	1	2
140	2	3
196	2	3

41.

EE.	*qr.*	*n.*
746	3	2
740	1	3

42.

FE.	*qr.*	*n.*
900	1	0
800	1	1

43.

yd.	*qr.*	*n.*
674	3	0
176	2	2

44.

Fr.E.	*qr.*	*n.*
504	1	3
246	5	0

82. SQUARE MEASURE.

45.

yd.	*f.*	*in.*
378	4	100
192	6	120
185	6	124

46.

a.	*r.*	*p.*
176	1	12
20	2	3

47.

yd.	*f.*	*in.*
100	8	1
100	1	20

48.

a.	*r.*	*p.*
814	1	2
701	0	20

49.

Sq. m.	*a.*	*r.*	*p.*
607	500	2	20
112	600	1	30

83. CUBIC MEASURE.

ROUGH TIMBER.

50.

ld.	*f.*	*in.*
237	1	0
108	26	1720
128	14	8

51.

ld.	*f.*	*in.*
301	2	1726
200	20	231

HEWN TIMBER.

52.

ld.	*f.*	*in.*
210	47	100
101	20	10

53.

ld.	*f.*	*in.*
471	12	200
199	49	700

54.

yd.	*f.*	*in.*
716	0	0
715	0	1727

84. WINE MEASURE.

55.

t.	*hhd.*	*gal.*	*qt.*	*pt.*	*gills.*
7	1	10	1	1	2
1	2	60	3	0	3
5	2	12	2	0	3

56.

gal.	*qt.*	*pt.*
37	1	1
31	2	0

57.

tier.	*gal.*	*qt.*
91	10	3
16	40	1

58.

a.	*gal.*	*qt.*
40	1	0
18	1	2

59.

qt.	*pt.*	*gills.*
79	1	1
1	1	2

85. BEER MEASURE.

60.

butts	*hhd.*	*gal.*	*qt.*
102	1	1	0
1	1	50	1
100	1	4	3

61.

kild.	*fir.*	*gal.*	*qt.*
11	1	2	1
10	1	3	0

62.

fir.	*gal.*	*qt.*
37	2	2
30	1	3

63.

gal.	*qt.*	*pt.*
78	0	0
77	0	1

64.

bar.	*gal.*	*qt.*
54	2	3
23	10	0

86. DRY MEASURE.

CORN.

65.

lasts	weys	qr.	bu.	pk.	gal.
24	1	2	1	0	1
12	1	4	7	1	0
11	1	2	1	3	1

66.

qr.	bu.	pk.
276	3	1
170	7	3

COALS.

67.

ch.	bu.	pk.
123	10	1
100	20	3

68.

sc.	ch.	sa.	bu.	pk.
9	11	2	1	3
4	20	8	2	1

87. TIME.

69.

mo.	w.	d.	h.	m.	"
101	1	2	3	4	5
100	1	3	5	7	9
0	3	5	21	56	56

70.

mo.	w.	d.	h.
123	3	5	11
123	1	6	13

71.

w.	d.	h.	m.
101	1	14	12
73	3	1	50

72.

d.	h.	m.	"
307	1	2	3
0	2	4	16

88. PROMISCUOUS EXAMPLES FOR PRACTICE.

73. Lent a tradesman 150*l.* 10*s.* and received of him in part 74*l.* 19*s.* 8*d.*½; what sum remains due to me? *Ans.* 75*l.* 10*s.* 3*d.*½.

74. Borrowed of a friend 1000*l.* and have since returned in part 234*l.* 5*s.* 6*d.*; what sum do I still owe him? *Ans.* 765*l.* 14*s.* 6*d.*

75. The amount of a person's property is 405*l.* 1*s.* 2*d.*¼, out of which he owes 178*l.* 11*s.* 9*d.*¾; what will he be worth when his debts are paid? *Ans.* 226*l.* 9*s.* 4*d.*½.

76. My yearly income is 567*l.* 12*s.* 8*d.* out of which I spend 390*l.* 18*s.* 1*d.*½; how much do I lay up? *Ans.* 176*l.* 14*s.* 6*d.*½.

77. An adventurer purchased a lottery ticket for 27*l.* 16*s.* 10*d.* his ticket came up a prize of 500*l.*; what sum did he gain? *Ans.* 472*l.* 3*s.* 2*d.*

78. A farmer carried seventy guineas to market, and brought $20l.\ 10s.\ 6d.\frac{1}{4}$ home; what sum did he spend? *Ans.* $52l.\ 19s.\ 5d.\frac{3}{4}$.

79. From a bar of silver, weighing *5lb.*, an artist contrived to file off unperceived a piece weighing *4oz. 3dwt. 2gr.*; what was the weight of the bar after this operation? *Ans. 4lb. 7oz. 16dwt. 22gr.*

80. A blacksmith bought 7 tons of iron, out of which he worked up during the first week *17cwt. 2qr.* $11\frac{1}{4}lb.$; what quantity had he remaining? *Ans. 6t. 2cwt. 1qr.* $16\frac{3}{4}lb.$

81. An apothecary mixed several drugs into a compound, weighing 3℔ 2℥ 1ʒ 2℈ 10*gr.* and after supplying his patients, he had 1℔ 5℥ 2ʒ 2℈ 12*gr.* of the compound left; what quantity did he send out? *Ans.* 1℔ 8℥ 6ʒ 2℈ 18*gr.*

82. A pedestrian engaged to travel 17 miles in 3 hours, on foot, but was obliged to give up at the distance of *4m. 5fur. 16p.* short of his journey's end; what distance did he travel? *Ans. 12m. 2fur. 24p.*

83. Through the middle of a park, containing 127 acres, a canal was cut, which occupied *19a. 3r. 37p.*; what quantity of land did the park contain after this was accomplished? *Ans. 107a. 3p.*

84. A shop-lifter purloined *14yds. 3qr. 1n.* of ribband from off a roll of 37 yards; what quantity did he leave? *Ans. 22yd. 3n.*

85. A ship of 200 tons burthen has already taken on board goods which occupy 38 cubic feet more than 158 tons; how much more is wanting to complete her lading? *Ans. 41t. 4f.*

86. A countryman had a rundlet of elder wine, and after treating his friends and neighbours, he bottled off the remainder, which was only *8gal. 2qt. 1pt. 3gills*; what quantity did the good folks drink? *Ans. 9gal. 1qt. 1gill.*

87 By the leaking of a beer-butt, which was full at the first, *12gal. 3qt.* $1\frac{1}{2}pt.$ was lost; what quantity remained in the butt after this accident? *Ans. 95gal.* $\frac{1}{2}pt.$

88 Two cart-horses thrust their heads into a bin, containing 3 quarters of oats, and had eaten, before they were detected, *3pk. 1gal. 2qt. 1pt.*; how much did they leave in the bin? *Ans. 2qr. 7bu. 1qt. 1pt.*

89. A watch, which usually goes 27 hours after winding up, stopped *8h. 13m.* 43″ short of the time; how long did the watch keep going? *Ans. 18h. 46m.* 17″.

COMPOUND MULTIPLICATION.

89. Compound Multiplication teaches how to multiply compound numbers by simple ones.

90. *When the multiplier does not exceed* 12.

RULE. Having written the multiplier under the lowest denomination of the multiplicand, multiply each denomination by it, beginning at the lowest, and reduce the several products as they arise to the next higher denomination; set down each remainder under its proper denomination, and carry the integers to the next succeeding product.

EXAMPLES.

91. MONEY[f].

1. Multiply 437*l.* 12*s.* 8*d.*$\frac{3}{4}$ by 5.

OPERATION.

	L.	s.	d.
	437	12	$8\frac{3}{4}$
			5
Prod.	2188	3	$7\frac{3}{4}$

Explanation.

I first multiply $\frac{3}{4}$ by 5; I say 5 times 3 are 15; 15 farthings are 3*d.*$\frac{3}{4}$; put down $\frac{3}{4}$, and carry 3. Next I say, 5 times 8 are 40 and 3 (carried) 43; 43 pence are 3*s.* 7*d.* put down 7, and carry 3. Next I say, 5 times 12 are 60 and 3 carried 63; 63 shillings are 3*l.* 3*s.* put down 3, and carry 3. Lastly, 5 times 7 are 35 and 3 carried 38; put down 8, and carry 3, &c. exactly as in simple Multiplication.

	2.			3.			4.		
	L.	s.	d.	L.	s.	d.	L.	s.	d.
Multiply	38	10	$9\frac{1}{4}$	27	8	$3\frac{1}{2}$	43	11	$7\frac{1}{4}$
By			4			6			7
Prod.	154	3	1	164	9	9	305	1	$2\frac{3}{4}$

---

[f] The reason of this rule is plain from the first example, as may be seen by multiplying each denomination separately, reducing the several products to the highest denomination possible, and adding all the results together.

			L.	s.	d.
Thus,	$\frac{3}{4} \times 5 = 15$ farthings	=	0	0	$3\frac{3}{4}$
	8*d.* $\times 5 = 40$ pence	=	0	3	4
	12*s.* $\times 5 = 60$ shillings	=	3	0	0
	437*l.* $\times 5 =$	=	2185	0	0
	These sums added give		2188	3	$7\frac{3}{4}$

as in the example; and the like may be shewn in every case.

5.			6.			7.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
78	16	2¾	91	13	1¼	123	4	5¼
		8			9			2
630	9	10	824	17	11¼			

8.			9.			10.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
371	7	6½	276	8	2¼	369	8	9¾
		3			4			5

11.			12.			13.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
807	9	3½	120	10	3¾	701	11	1¼
		6			7			8

14.			15.			16.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
613	12	10½	800	9	8¾	243	10	2¼
		9			10			11

17.			18.			19.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
139	11	8¼	296	10	10½	367	12	5¾
		4			5			6

20.			21.			22.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
419	8	11	536	9	10¼	674	10	11
		7			8			9

23.			24.			25.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
760	0	2½	817	12	0¾	901	2	1¼
		10			11			12

92. TROY WEIGHT.

26. Multiply 4*lb.* 5*oz.* 6*dwt.* 7*gr.* by 7.

OPERATION.

lb.	*oz.*	*dwt.*	*gr.*
4	5	6	7
			7
31	1	4	1

Explanation.

Here I say, 7 times 7 are 49 grains, which make 2 dwts. and 1 grain over; put down 1, and carry 2; then 7 times 6 are 42 and 2 carried 44 dwts. which make 2oz. and 4 dwts. over; put down 4, and carry 2; next 7 times 5 are 35 and 2 carried 37 ounces, which make 3 lb. and 1 oz. over; put down 1 and carry 3; lastly, 7 times 4 are 28 and 3 carried 31; put it down.

27.

lb.	*oz.*	*dwt.*	*gr.*
7	11	18	10
			2
15	11	16	20

28.

lb.	*oz.*	*dwt.*	*gr.*
1	2	3	4
			3

29.

lb.	*oz.*	*dwt.*	*gr.*
2	3	4	5
			4

30.

lb.	*oz.*	*dwt.*	*gr.*
3	4	5	6
			5

31.

lb.	*oz.*	*dwt.*	*gr.*
4	10	12	11
			6

93. APOTHECARIES' WEIGHT.

32. Multiply 5*lb.* 10℥ 7ʒ 2℈ 8*gr.* by 8.

OPERATION.

lb.	℥	ʒ	℈	*gr.*
5	10	7	2	8
				8
47	3	6	1	4

Explanation.

First, 8 times 8 are 64, or 3 scruples and 4 over to put down; then 8 times 2 are 16 and 3 are 19 scruples, or 6 drams and 1 scruple over to put down; then 8 times 7 are 56 and 6 are 62 drams, or 7 ounces and 6 drams over to put down; next 8 times 10 are 80 and 7 are 87 ounces, or 7 pounds, and 3 ounces over to put down; lastly, 8 times 5 are 40 and 7 are 47 to put down.

33.

lb.	℥	ʒ	℈	*gr.*
8	7	6	1	18
				4
34	7	2	1	12

34.

lb.	℥	ʒ	℈
9	1	2	1
			5

35.

℥	ʒ	℈	*gr.*
1	3	2	5
			6

36.

lb.	℥	ʒ	℈
4	8	6	1
			7

37.

lb.	℥	ʒ	℈	*gr.*
7	2	4	0	10
				8

94. AVOIRDUPOIS WEIGHT.

38.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
6	7	3	8	9	10
					5
31	19	0	15	0	2

39.

t.	*cwt.*	*qr.*	*lb.*
29	8	2	9
			2

40.

cwt.	*qr.*	*lb.*	*oz.*
14	1	2	3
			3

41.

cwt.	*qr.*	*lb.*	*oz.*	*dr.*
9	3	20	11	12
				4

95. LONG MEASURE.

42.

lea.	*m.*	*fur.*	*p.*
2	2	4	10
			6
17	0	1	20

43.

yd.	*f.*	*in.*	*bc.*
4	1	3	2
			3

44.

cub.	*in.*	*l.*
27	13	11
		4

45.

f.	*in.*	*l.*
10	11	3
		5

46.

yd.	*f.*	*in.*	*bc.*
17	2	10	1
			6

96. CLOTH MEASURE.

47.

yd.	*qr.*	*n.*
28	3	2
		8
231	0	0

48.

yds.	*qr.*	*n.*
241	1	3
		4

49.

EE.	*qr.*	*n.*
139	4	0
		5

50.

FE.	*qr.*	*n.*
163	2	1
		6

51.

Fr.E.	*qr.*	*n.*
273	5	3
		7

97. SQUARE MEASURE.

52.

sq.m.	*a.*	*r.*	*p.*
23	30	1	20
			9
207	273	1	20

53.

a.	*r.*	*p.*
137	2	10
		5

54.

yd.	*f.*	*in.*
375	8	100
		6

55.

a.	r.	p.
741	3	16
		7

56.

sq.m.	a.	r.	p.
47	400	1	12
			8

98. CUBIC MEASURE.

ROUGH TIMBER.

57.

ld.	f.	in.
24	20	1000
		10
245	5	1360

58.

ld.	f.	in.
32	1	85
		6

HEWN TIMBER.

59.

ld.	f.	in.
29	40	144
		7

60.

ld.	f.	in.
71	48	1234
		8

61.

yd.	f.	in.
38	20	1600
		9

99. WINE MEASURE.

62.

t.	hhd.	gal.	qt.	pt.	gills
21	3	60	3	1	3
					11
241	3	40	2	1	1

63.

hhd.	gal.	qt.
12	5	2
		7

64.

t.	hhd.	gal.
10	1	12
		8

65.

a.	gal.	qt.
27	4	3
		9

66.

tier.	gal.	qt.
57	30	1
		10

100. BEER MEASURE.

67.

butts	hhd.	gal.	qt.	pt.
72	1	10	2	1
				12
871	0	19	2	0

68.

fir.	gal.	qt.
13	4	1
		8

69.

gal.	qt.	pt.
32	1	1
		9

70.			71.			
bar.	*gal.*	*qt.*	*hhd.*	*fir.*	*gal.*	*qt.*
75	12	3	21	1	0	2
		10				11

101. DRY MEASURE.

72.

lasts	*weys*	*qr.*	*bu.*	*pk.*	*gal.*
10	1	3	5	2	1
					7
76	0	0	7	1	1

73.			74.		
qr.	*bu.*	*pk.*	*lasts*	*weys*	*qr.*
231	1	2	102	1	4
		3			4

75.			76.		
bu.	*pk.*	*gal.*	*ch.*	*bu.*	*pk.*
371	3	1	407	12	3
		5			6

102. TIME.

77.

mo.	*w.*	*d.*	*h.*	*m.*	*″*
7	3	4	8	9	12
					8
63	0	6	17	13	36

78.			79.		
h.	*m.*	*″*	*mo.*	*w.*	*d.*
71	12	3	36	2	1
		9			10

80.			81.			
w.	*d.*	*h.*	*d.*	*h.*	*m.*	*″*
10	6	12	37	14	50	13
		11				12

103. *When the multiplier is a composite number.*

RULE. Multiply the given compound number by one of the component parts, and that product by the other.

To prove the operation, change the order of the multipliers, that is, multiply by that part *first* which you multiplied by *last* in the preceding work, and by that *last* which you multiplied by *first* [s].

[s] This rule is evident from what has been said in the note on the corresponding rule in simple Multiplication, Art. 35.

82. Multiply 2*l*. 3*s*. 4*d*.¼ by 20, several ways.

1st method by 4×5.			2d method by 5×4.			3d method by 2×10.			4th method by 10×2.		
L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
2	3	4¼	2	3	4¼	2	3	4¼	2	3	4¼
		4			5			2			10
8	13	5	10	16	9¼	4	6	8½	21	13	6½
		5			4			10			2
Prod. 43	7	1	43	7	1	43	7	1	43	7	1

83. Multiply 5*t*. 4*cwt*. 3*qr*. 2*lb*. 1*oz*. 6*dr*. by 24, six ways.

1st method by 4×6.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					4
20	19	0	8	5	8
					6
Prod. 125	14	1	22	1	0

2d method by 6×4.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					6
31	8	2	12	8	4
					4
125	14	1	22	1	0

3d method by 3×8.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					3
15	14	1	6	4	2
					8
Prod. 125	14	1	22	1	0

4th method by 8×3.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					8
41	18	0	16	11	0
					3
125	14	1	22	1	0

5th method by 2×12.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					2
10	9	2	4	2	12
					12
Prod. 125	14	1	22	1	0

6th method by 12×2.

t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	4	3	2	1	6
					12
62	17	0	25	0	8
					2
125	14	1	22	1	0

84. Multiply 12*l*. 1*s*. 2*d*.¼ by 14. *Prod.* 168*l*. 16*s*. 7*d*.½.
85. Multiply 21*l*. 2*s*. 3*d*.½ by 32. *Prod.* 675*l*. 13*s*. 4*d*.
86. Multiply 1*l*. 16*s*. 9*d*.¾ by 42. *Prod.* 77*l*. 6*s*. 1*d*.½.
87. Multiply 7*s*. 9*d*.½ by 108. *Prod.* 42*l*. 1*s*. 6*d*.
88. Multiply 2*l*. 16*s*. 10*d*.¾ by 144. *Prod.* 409*l*. 13*s*.

70.				71.			
bar.	*gal.*	*qt.*		*kild.*	*fir.*	*gal.*	*qt.*
75	12	3		21	1	0	2
		10					11

101. DRY MEASURE.

72.

lasts	*weys*	*qr.*	*bu.*	*pk.*	*gal.*
10	1	3	5	2	1
					7
76	0	0	7	1	1

73.

qr.	*bu.*	*pk.*
231	1	2
		3

74.

lasts	*weys*	*qr.*
102	1	4
		4

75.

bu.	*pk.*	*gal.*
371	3	1
		5

76.

ch.	*bu.*	*pk.*
407	12	3
		6

102. TIME.

77.

mo.	*w.*	*d.*	*h.*	*m.*	″
7	3	4	8	9	12
					8
63	0	6	17	13	36

78.

h.	*m.*	″
71	12	3
		9

79.

mo.	*w.*	*d.*
36	2	1
		10

80.

w.	*d.*	*h.*
10	6	12
		11

81.

d.	*h.*	*m.*	″
37	14	50	13
			12

103. *When the multiplier is a composite number.*

Rule. Multiply the given compound number by one of the component parts, and that product by the other.

To prove the operation, change the order of the multipliers, that is, multiply by that part *first* which you multiplied by *last* in the preceding work, and by that *last* which you multiplied by *first* s.

s This rule is evident from what has been said in the note on the corresponding rule in simple Multiplication, Art. 35.

82. Multiply 2*l.* 3*s.* 4*d.*$\frac{1}{4}$ by 20, several ways.

	1st method by 4×5.			2d method by 5×4.			3d method by 2×10.			4th method by 10×2.		
	L.	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*	*L.*	*s.*	*d.*
	2	3	$4\frac{1}{4}$	2	3	$4\frac{1}{4}$	2	3	$4\frac{1}{4}$	2	3	$4\frac{1}{4}$
			4			5			2			10
	8	13	5	10	16	$9\frac{1}{4}$	4	6	$8\frac{1}{2}$	21	13	$6\frac{1}{2}$
			5			4			10			2
Prod.	43	7	1	43	7	1	43	7	1	43	7	1

83. Multiply 5*t.* 4*cwt.* 3*qr.* 2*lb.* 1*oz.* 6*dr.* by 24, six ways.

	1st method by 4×6.						2d method by 6×4.					
	t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*	*t.*	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
	5	4	3	2	1	6	5	4	3	2	1	6
						4						6
	20	19	0	8	5	8	31	8	2	12	8	4
						6						4
Prod.	125	14	1	22	1	0	125	14	1	22	1	0

	3d method by 3×8.						4th method by 8×3.					
	t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*	*t.*	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
	5	4	3	2	1	6	5	4	3	2	1	6
						3						8
	15	14	1	6	4	2	41	18	0	16	11	0
						8						3
Prod.	125	14	1	22	1	0	125	14	1	22	1	0

	5th method by 2×12.						6th method by 12×2.					
	t.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*	*t.*	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
	5	4	3	2	1	6	5	4	3	2	1	6
						2						12
	10	9	2	4	2	12	62	17	0	25	0	8
						12						2
Prod.	125	14	1	22	1	0	125	14	1	22	1	0

84. Multiply 12*l.* 1*s.* 9*d.*$\frac{1}{4}$ by 14. *Prod.* 168*l.* 16*s.* 7*d.*$\frac{1}{2}$.
85. Multiply 21*l.* 2*s.* 3*d.*$\frac{1}{2}$ by 32. *Prod.* 675*l.* 13*s.* 4*d.*
86. Multiply 1*l.* 16*s.* 9*d.*$\frac{3}{4}$ by 42. *Prod.* 77*l.* 6*s.* 1*d.*$\frac{1}{2}$.
87. Multiply 7*s.* 9*d.*$\frac{1}{2}$ by 108. *Prod.* 42*l.* 1*s.* 6*d.*
88. Multiply 2*l.* 16*s.* 10*d.*$\frac{3}{4}$ by 144. *Prod.* 409*l.* 13*s.*

102. Multiply 4*l.* 3*s.* 2*d.*$\frac{1}{4}$ by 2345.

OPERATION.

L.	s.	d.	
4	3	2$\frac{1}{4}$ × 5	
		10	
41	11	10$\frac{1}{2}$ × 4	
		10	
415	18	9 × 3	
		10	
4159	7	6	
		2	
8318	15	0	= *prod. by* 2000
1247	16	3	= 300
166	7	6	= 40
20	15	11$\frac{1}{4}$	= 5
9753	14	8$\frac{1}{4}$	= *prod. by* 2345

Explanation.

Here the highest place of the multiplier is thousands; I therefore multiply by 10 × 10 × 10 for 1000: I next multiply this product by 2 for 2000: I next multiply the product of 100 (or 10 × 10) by 3 for the product of 300; this I place under the former: then I multiply the product of 10 by 4, which gives the product of 40; this I place under the others: I next multiply the top line by 5, and place the product under the former ones. Lastly, I add the four products together for the answer.

103. Multiply 1*s.* 3*d.*$\frac{1}{2}$ by 432.

	L.	s.	d.
	0	1	3$\frac{1}{2}$ × 2
			10
	0	12	11 × 3
			10
	6	9	2
			4
	25	16	8
	1	18	9
	0	2	7
Product	27	18	0

104. Multiply 1*yd.* 2*qr.* 3*n.* by 1068.

	yd.	qr.	n.
	1	2	3 × 8
			10
	16	3	2 × 6
			10
	168	3	0
			10
	1687	2	0
	101	1	0
	13	2	0
Product	1802	1	0

105. Multiply 4*l.* 9*s.* 6*d.* by 156. *Prod.* 698*l.* 2*s.*

106. Multiply 17*s.* 5*d.*¼ by 394. *Prod.* 343*l.* 18*s.* 7*d.*

107. Multiply 1*s.* 8*d.*¼ by 3016. *Prod.* 257*l.* 12*s.* 4*d.*

108. Multiply 1*cwt.* 2*qr.* 3*lb.* by 4321. *Prod.* 6597*cwt.* 0*qr.* 27*lb.*

109. Multiply 1*tun,* 3*hhd.* 2*gal.* 1*qt.* by 980. *Prod.* 1723 *tuns,* 3*hhd.*

106. PROMISCUOUS EXAMPLES FOR PRACTICE.

110. Seven tailors received each 1*l.* 4*s.* 9*d.* for a week's wages; what sum was sufficient to pay them? *Ans.* 8*l.* 13*s.* 3*d.*

111. What is the value of 16 cwt. of sugar, at 3*l.* 17*s.* 4*d.* per cwt.? *Ans.* 61*l.* 17*s.* 4*d.*

112. Bought 120 dozen of candles, at 11*s.* 6*d.* per dozen; what sum do they amount to? *Ans.* 69*l.*

113. Sold 96 gallons of rum, at 1*l.* 8*s.* 6*d.* per gallon; what sum will pay for the whole? *Ans.* 136*l.* 16*s.*

114. What is the worth of 17 yards of velvet, at 1*l.* 3*s.* 1*d.*¼ per yard? *Ans.* 19*l.* 13*s.* 1*d.*¼

115. Required the weight of 1000 pieces of gold coin, each weighing 6*dwts.* 7*gr.* *Ans.* 26*lb.* 2*oz.* 11*dwt.* 16*gr.*

116. What is the weight of 19 chests of tea, each 1*cwt.* 0*qr.* 14*lb.*? *Ans.* 21*cwt.* 1*qr.* 14*lb.*

117. To fill a cooler, there were put in 105 pails of liquor, each 3*gal.* 1*qt.* 1*pt.*; what quantity did the cooler hold? *Ans.* 6*hhd.* 30*gal.* 1*qt.* 1*pt.*

118. A bankrupt owes in all 2468*l.* and can pay 15*s.* 6*d.*¼ in the pound; what are his effects worth? *Ans.* 1917*l.* 16*s.* 10.

119. A detachment, consisting of 3258 cavalry, being sent on a particular service, during which each horse ate 3*bu.* 3*pk.* 1*gal.* of oats; how many quarters did the whole detachment consume? *Ans.* 1578*qr.* 3*pk.*

COMPOUND DIVISION.

107. Compound Division teaches how to divide compound numbers by simple ones, that is, to divide a compound number into any assigned number of equal parts.

108. *When the divisor does not exceed* 12.

RULE. Place the divisor to the left hand of the dividend.

Divide the highest denomination of the dividend, and set the quotient under, as in simple Division. Reduce the remainder (if there is any) to the next inferior name, and add to it the number which is of the same name in the dividend. Divide, set down the quotient, and reduce the remainder to the next inferior name; proceed in this manner until you have divided all the denominations in the given dividend[i].

The method of proof. Multiply the quotient by the divisor, and add in the remainder; the result will be like the dividend, if the work is right.

109. Examples in Money.

1. Divide 13570*l.* 1*s.* 3*d.*$\frac{3}{4}$ by 6.

OPERATION.

	L.	*s.*	*d.*
	6)13570	1	$3\frac{3}{4}$(3
Quot.	2261	13	$6\frac{1}{2}$
Proof	13570	1	$3\frac{3}{4}$

Explanation.

The pounds are divided as in simple Division, (Art. 37. B.) after which there are 4 remaining; therefore I say, 4 pounds are 80 shillings and 1 are 81; sixes in 81 will go 13 times, and 3 over; put down 13, and reduce the 3 over into pence; thus, 3 shillings are 36 pence and 3 (in the dividend to add in) are 39; sixes in 39 will go 6 times, (to put down,) and 3 over: 3 pence are 12 farthings and 3 (in the dividend) are 15; sixes in 15 will go twice, (put down 2 or $\frac{1}{2}$,) and 3 over. I multiply the quotient by 6, (Art. 91.) and the result is the proof.

2.

L.	*s.*	*d.*
4)381	3	$5\frac{1}{4}$(1
95	5	$10\frac{1}{4}$
381	3	$5\frac{1}{4}$

3.

L.	*s.*	*d.*
5)739	2	$3\frac{1}{2}$(
147	16	$5\frac{1}{2}$
739	2	$3\frac{1}{2}$

4.

L.	*s.*	*d.*
7)133	14	$8\frac{3}{4}$(
19	2	$1\frac{1}{4}$
133	14	$8\frac{3}{4}$

5.

L.	*s.*	*d.*
8)100	1	$2\frac{1}{4}$(1
12	10	$1\frac{3}{4}$
100	1	$2\frac{1}{4}$

6.

L.	*s.*	*d.*
2)357	1	$3\frac{1}{4}$(

7.

L.	*s.*	*d.*
3)417	10	$7\frac{1}{4}$(

---

[i] To divide a number consisting of several denominations by any simple number, is evidently no more than to divide the several parts of the former by the latter: if any denomination be not exactly divisible, it is plain we must divide as much of it as will exactly divide, reduce the rest to the next lower denomination, and proceed as directed in the rule; whence, since every part of the dividend is divided, the several results collected will form the quotient.

8.	9.	10.
L. s. d.	L. s. d.	L. s. d.
4)987 6 5¾(	5)702 12 11½(	6)213 10 10¾(

11.	12.	13.
L. s. d.	L. s. d.	L. s. d.
7)121 9 2½(	8)372 1 3¼(	9)210 12 4½(

14.	15.	16.
L. s. d.	L. s. d.	L. s. d.
2)103 0 5½(	3)140 10 0¼(	4)237 1 11½(

17.	18.	19.
L. s. d.	L. s. d.	L. s. d.
5)302 12 6(	6)700 9 6¾(	7)301 18 10(

20.	21.	22.
L. s. d.	L. s. d.	L. s. d.
8)517 13 1¾(	9)739 17 8¼(	9)102 10 10½(

23.	24.	25.
L. s. d.	L. s. d.	L. s. d.
10)654 11 8(	11)376 1 2¾(	12)781 10 3¼(

110. TROY WEIGHT.

26. Divide 235*lb.* 3*oz.* 2*dwt.* 12*gr.* by 7.

OPERATION.

	lb.	oz.	dwt.	gr.
7)	235	3	2	12(5
Quot.	33	7	6	1
Proof	235	3	2	12

Explanation.

Having divided the pounds, there are 4 over; I therefore say, 4 pounds are 48 ounces and 3 are 51; sevens in 51 will go 7 times, and 2 over; put down 7; then 2 ounces are 40 dwts. and 2 are 42; sevens in 42 will go 6 times, and nothing over; put down 6, and there is nothing to carry; lastly, sevens in 12 will go once, and 5 over; put down 1, and 5 to the right hand for a remainder. In the proof the 5 is taken in with the grains; the rest is obvious.

27.

	lb.	oz.	dwt.	gr.
5)	37	1	3	5(2
	7	5	0	15
	37	1	3	5

28.

	lb.	oz.	dwt.	gr.
2)	10	11	13	11(

29.

	lb.	oz.	dwt.	gr.
3)	71	1	9	2(

30.

	lb.	oz.	dwt.	gr.
4)	81	10	17	21(

111. APOTHECARIES' WEIGHT.

31 Divide 137℔ 1℥ 2ʒ 1℈ 10*gr.* by 8.

OPERATION.

	℔.	℥	ʒ	℈	gr.
8)	137	1	2	1	10(6
	17	1	5	0	18
	137	1	2	1	10

Explanation.

After dividing the lbs., 1 remains; 1 pound is 12 ounces and 1 are 13; eights in 13 will go once, and 5 over; put down 1; then 5 ounces are 40 drams and 2 are 42; eights in 42 will go 5 times, and 2 over; put down 5; then 2 drams are 6 scruples and 1 are 7; eights in 7 will not go; therefore put down 0; 7 scruples are 140 grains and 10 are 150; eights in 150 will go 18 times, and 6 over.

32.

	℔.	℥	ʒ	℈	gr.
6)	25	2	3	1	2(4
	4	2	3	0	13
	25	2	3	1	2

33.

	lb.	℥	ʒ	℈	gr.
3)	17	1	1	1	4(

34.

	lb.	℥	ʒ	℈	gr.
4)	71	2	3	0	8(

35.

	lb.	℥	ʒ	℈	gr.
5)	47	10	1	2	12(

112. AVOIRDUPOIS WEIGHT.

36.

	t.	cwt.	qr.	lb.	oz.	dr.
5)	34	1	2	3	4	5(
	6	16	1	6	4	1
	34	1	2	3	4	5

37.

	t.	cwt.	qr.	lb.
2)	11	3	3	7(

38.

	cwt.	qr.	lb.	oz.
3)	17	3	10	11(

39.

	qr.	lb.	oz.	dr.
4)	59	12	11	10(

113. LONG MEASURE.

40.

	yds.	f.	in.	l.
6)	70	2	10	7(1
	11	2	5	9
	70	2	10	7

41.

	lea.	m.	fur.	p.
3)	54	2	3	12(

42.

	yds.	f.	in.	bc.
4)	97	1	11	1(

43.

	lea.	m.	fur.	p.
5)	14	0	1	24(

114. CLOTH MEASURE.

44.

	EE.	qr.	n.
7)	591	1	3(4
	84	2	1
	591	1	3

45.

	yds.	qr.	n.
4)	1875	3	1(

46.

	FE.	qr.	n.
5)	1234	2	2(

47.

	Fr.E.	qr.	n.
6)	9017	5	0(

115. SQUARE MEASURE.

48.

	sq. m.	a.	r.	p.
8)	305	1	2	3(3
	38	80	0	30
	305	1	2	3

49.

	a.	r.	p.
5)	3714	3	12(

50.

	a.	r.	p.
6)	7941	1	20(

51.

	yd.	f.	in.
7)	1234	3	123(

116. CUBIC MEASURE.

ROUGH TIMBER.

52.

	ld.	f.	in.
5)	7142	1	1000(3
	1428	16	545
	7142	1	1000

53.

	ld.	f.	in.
2)	7131	2	100(

HEWN TIMBER.

54.

	ld.	f.	in.
3)	5714	11	300(

55.

	yd.	f.	in.
4)	1232	40	1234(

117. WINE MEASURE.

56.

	t.	hhd.	gal.	qt.	pt.	gills
6)	71	1	12	3	1	3(1
	11	3	33	2	1	1
	71	1	12	3	1	3

57.

	t.	hhd.	gal.	qt.
3)	79	3	12	2(

58.

	tier.	gal.	qt.
4)	271	20	0(

59.

	a.	gal.	qt.	pt.
5)	371	9	2	1(

118. BEER MEASURE.

60.

	butts	hhd.	gal.	qt.	
9)	123	1	12	3	(6
	13	1	25	1	
	123	1	12	3	

61.

	bar.	gal.	qt.	pt.	
6)	514	1	2	1	(

62.

	kild.	fir.	gal.	qt.	
7)	200	1	0	1	(

63.

	fir.	gal.	qt.	pt.	
8)	37	3	2	0	(

119. DRY MEASURE.

64.

	lasts	wey	qr.	bu.	pk.	
11)	271	1	4	7	3	(7
	24	1	2	2	0	
	271	1	4	7	3	

65.

	qr.	bu.	pk.	
8)	741	3	2	(

66.

	ch.	bu.	pk.	
9)	123	12	3	(

67.

	ch.	sa.	bu.	pk.	
10)	123	1	2	3	(

120. TIME.

68.

	mo.	w.	d.	h.	m.	"	
9)	317	1	2	3	4	5	(2
	35	1	0	5	40	27	
	317	1	2	3	4	5	

69.

	mo.	w.	d.	
10)	7351	3	6	(

70.

	w.	d.	h.	
11)	209	1	12	(

71.

	d.	h.	m.	"	
12)	31	10	12	20	(

121. *When the divisor is a composite number.*

RULE. Divide the given compound number by one of the component parts, and that quotient by the other, the last quotient will be the answer, when there are only 2 component parts; if there are more than two, divide successively by them all, and the last quotient will be the answer.

To prove the operation, change the order of the divisors as in Art. 103.

72. Divide 37*l.* 12*s* 5*d.*½ by 16, several ways.

1st method
by 4×4.

	L.	*s.*	*d.*	
4	37	12	5½	2
4	9	8	1¼	1
Quot.	2	7	0¼	

Rem. 1×4+2=6.

2d method
by 8×2.

	L.	*s.*	*d.*	
8	37	12	5½	6
2	4	14	0½	0
	2	7	0¼	

Rem. 6.

3d method
by 2×8.

	L.	*s.*	*d.*	
2	37	12	5½	0
8	18	16	2¾	3
	2	7	0¼	

Rem. 3×2=6.

73. Divide 300*l.* 10*s.* 6*d.*¾ by 210, or 3×7×10.

1st method
by 3×10×7.

	L.	*s.*	*d.*	
3	300	10	6¾	0
10	100	3	6¼	9
7	10	0	4	5
Quot.	1	8	7¼	

Rem. $\overline{5\times10+9}\times3=177$.

2d method
by 3×7×10.

	L.	*s.*	*d.*	
3	300	10	6¾	0
7	100	3	6¼	3
10	14	6	2½	8
	1	8	7¼	

Rem. $\overline{8\times7+3}\times3=177$.

3d method
by 10×7×3.

	L.	*s.*	*d.*	
10	300	10	6¾	7
7	30	1	0½	3
3	4	5	10¼	2
Quot.	1	8	7¼	

Rem. $\overline{2\times7+3}\times10+7=177$.

74. Divide 168*l.* 16*s.* 7*d.*¼ by 14. *Quot.* 12*l.* 1*s.* 2*d.*¼.

75. Divide 42*l.* 1*s.* 6*d.* by 54. *Quot.* 0*l.* 15*s.* 7*d.*

76 Divide 409*l.* 18*s.* by 144. *Quot.* 2*l.* 16*s.* 10*d.*¼.

77. Divide 675*l.* 5*s.* 5*d.*¼ by 240. *Quot.* 2*l.* 16*s.* 3*d.*¼ 22 *rem.*

78. Divide 19*lb.* 0*oz.* 19*dwt.* 9*gr.* by 15. *Quot.* 1*lb.* 3*oz.* 6*dwt.* 7*gr.*

79. Divide 128*tuns*, 1*hhd.* 60*gal.* 3*qt.* by 27. *Quot.* 4*tuns*, 3*hhd.* 2*gal.* 1*qt.*

80. Divide 1185*qr.* 7*bu.* 2*pk.* 0*gal.* by 132. *Quot.* 8*qr.* 7*bu.* 3*pk.* 1*gal.*

122. *When the divisor consists of a number with ciphers to the right hand.*

RULE. Divide by the significant figures only, without regarding the ciphers, and from the highest denomination of the quotient, cut off by a small line as many of its right hand figures as there are ciphers in the divisor. Reduce these right hand figures to the next lower denomination, taking in the figures of the same denomination from the said quotient. Cut off from the right of this number as many figures as there are ciphers in the divisor, reduce these to the next lower denomination, cut off, &c. and proceed in this manner through all the denominations; the numbers on the left hand of the cutting off lines being placed in order will be the quotient required.

81. Divide 37921*l.* 2*s.* 8*d.*¼ by 9000.

OPERATION.

```
    L.    s.  d.
9)37921   2   8¼(
  ----------------
  4|213   9   2¼
     20
  -----
  4|269
     12
  -----
  3|230
      4
  -----
  0|921
```

Explanation.

I first divide by 9, neglecting the ciphers; then, because there are 3 ciphers, I cut off 3 figures, viz. 213, from the left of the pounds; this number I reduce to shillings, taking in the 9; from the shillings in like manner I cut off 3 figures, viz. 269, which I reduce into pence, taking in the 2; from the result I cut off 230; which I reduce into farthings, taking in the 1; and cut off 921, which is the remainder, and the left hand numbers cut off placed in order constitute the quotient.

Quotient 4*l.* 4*s.* 3*d.* 921 *rem.*

82. Divide 37237*d.* 6*h.* 6*m.* 24″. by 1200.

```
      d.     h.  m.   "
12)37237     6   6   24(
   ---------------------
   31|03     2  30   32
      24
    ----
     |74
      60
   -----
   44|70
      60
   -----
   42|32
```

Quotient 31*d.* 0*h.* 44*m.* 42″. 32 *rem.*

83. Divide 670*l.* 12*s.* 6*d.* by 600. *Quot.* 1*l.* 2*s.* 4*d.*$\frac{1}{5}$.

84. Divide 332*l.* 15*s.* by 80. *Quot.* 4*l.* 3*s.* 2*d.*$\frac{1}{4}$.

85. Divide 54321*yd.* 3*qr.* 1*n.* by 11000. *Quot.* 4*yd.* 3*qr.* 3*n.* 13 *rem.*

86. Divide 158*tons,* 11*cwt.* 2*qr.* 13*lb.* by 100. *Quot.* 1*ton,* 11*cwt.* 2*qr.* 24*lb.* 21 *rem.*

123. *When the divisor is any number greater than* 12.

RULE. Divide the highest denomination, and place the quotient on the right hand, exactly like common Long Division. Reduce the remainder (if any) to the next lower denomination, taking in the figures which are of the same denomination. Divide this number, the quotient will be of the same denomination with it. Reduce the remainder, take in, divide, &c. until the whole is finished.

87. Divide 12345*l.* 6*s.* 7*d.*$\frac{3}{4}$ by 215.

OPERATION.

```
      L.   s. d.   L. s. d.
215)12345  6  7¾(57  8  4¾
    1075
    ----
     1595
     1505
     ----
       90
       20
     ----
     1806
     1720
     ----
       86
       12
     ----
     1039
      860
     ----
      179
        4
     ----
      719
      645
     ----
       74
```

Explanation.

Having divided the pounds, the quotient is 57, and the remainder 90; this latter I multiply by 20, and take in the 6; I then divide the result, viz. 1806, and the quotient 8 is shillings; the remainder 86 I multiply by 12, and take in the 7; the result, viz. 1039, I divide, and the quotient 4 is pence; the remainder 179 I multiply by 4, and take in the 3; the result I divide, and the quotient 3 is farthings.

88. Divide 71236*yds.* 3*qr.* 2*n.* by 1234.

```
         yd.   qr.  n.  yd. qr. n.
1234)71236   3   2(57  2  3 Quotient.
     6170
     ----
      9536
      8638
      ----
       898
         4
      ----
      3595
      2468
      ----
      1127
         4
      ----
      4510
      3702
      ----
       808
      ----
```

89. Divide 39*l.* 1*s.* 5*d.*$\frac{1}{2}$ by 31. *Quot.* 1*l.* 5*s.* 2*d.*$\frac{1}{2}$.

90. Divide 412*l.* 1*s.* 9*d.* by 111. *Quot.* 3*l.* 14*s.* 3*d.*

91. Divide 9753*l.* 14*s.* 8*d.*$\frac{1}{2}$ by 2345. *Quot.* 4*l.* 3*s.* 2*d.*$\frac{1}{4}$

92. Divide 177*kild.* 1*fir.* 6*gal.* 1*qt.* by 65. *Quot.* 2*kild.* 1*fir.* 4*gal.* 1*qt.*

124. PROMISCUOUS EXAMPLES FOR PRACTICE.

93. If I pay 9*l.* 9*s.* for 8 lb. of tea, what is the price per lb.? *Ans.* 1*l.* 3*s.* 7*d.*$\frac{1}{2}$.

94. Paid 13*l.* 18*s.* 8*d.* being the week's wages of 11 carpenters; what sum did each receive? *Ans.* 1*l.* 5*s.* 4*d.*

95. Bought 12 pigs for 10*l.* 12*s.* 6*d.* what is the value of each? *Ans.* 17*s.* 8*d.*$\frac{1}{2}$.

96. Fourteen gentlemen hire a yacht or pleasure-boat, the expences of which will amount to 40*l.* 9*s.* 4*d.*$\frac{1}{2}$; what will each have to pay? *Ans.* 2*l.* 17*s.* 9*d.*$\frac{3}{4}$.

97. If 17 gallons of brandy cost 18*l.* 18*s.* 3*d.* how much is that per gallon? *Ans.* 1*l.* 2*s.* 3*d.*

98. A club of 39 persons divide a lottery-prize of 20000*l.* equally; how much does each receive? *Ans.* 512*l.* 16*s.* 4*d.*$\frac{3}{4}$, 27 *rem.*

99. A farm of 173 acres was reaped by 72 persons; how many acres is that apiece? *Ans.* 2*a.* 1*r.* 24*p.* 32 *rem.*

100. Suppose 7 puncheons of rum are just sufficient to serve 123 sailors during a voyage, how much may each man drink? *Ans.* 4*gal.* 3*qt.* 15 *rem.*

101. If five thousand fathoms of rope be made up into coils of 117 fathoms each, how many coils will there be? *Ans.* *42 coils and 86 fathoms over.*

102. How much can a person, whose income is a thousand a year, afford to spend per day? *Ans.* 2*l.* 14*s.* 9*d.*$\frac{1}{4}$, 50 *rem.*

103. If a greyhound in going over a mile of ground make 1537 leaps, what is the length of each leap? *Ans.* 3*f.* 5*in.* 343 *rem.*

104. If a pipe of wine cost 120*l.* how much will be the charge per dozen, supposing a dozen equal to 3 gallons? *Ans.* 2*l.* 17*s.* 1*d.*$\frac{1}{4}$, 36 *rem.*

105. A silversmith, out of 23*lb.* 9*oz.* 6*dwt.* of silver, made 9 dozen of spoons; required the weight of each? *Ans.* 2*oz.* 12*dwt.* 20*gr.*

125. *The following questions require both multiplication and division.*

RULE. To multiply by $\frac{1}{2}$ you must divide by 2; to multiply by $\frac{1}{4}$, divide by 4; and to multiply by $\frac{3}{4}$, divide by 2, and that quotient by 2, and add both quotients together.

106. What is the value of 3$\frac{1}{2}$ lb. of tea, at 12*s.* 9*d.* per lb.? *Ans.* 2*l.* 4*s.* 7*d.*$\frac{1}{2}$.

Multiply the top line by 3, divide it by 2, and add both results together.

107. What will 8$\frac{1}{4}$ cwt. of cheese cost, at 4*l.* 4*s.* 6*d.* per cwt.? *Ans.* 34*l.* 17*s.* 1*d.*$\frac{1}{2}$.

Multiply by 8, divide (the top line) by 4, and add both results together.

108. What will 12$\frac{3}{4}$ dozen of wine cost, at 2*l.* 10*s.* per dozen? *Ans.* 31*l.* 17*s.* 6*d.*

Multiply by 12, divide (the top line) by 2, and this last result by 2, then add all the three results together.

109. Required the value of 25$\frac{1}{2}$ yards of cloth, at 3*s.* 4*d.*$\frac{1}{2}$ per yard? *Ans.* 4*l.* 6*s.* 0*d.*$\frac{3}{4}$.

110. What must be given for 117$\frac{1}{4}$ lb. of tea, at 12*s.* 6*d.* per lb.? *Ans.* 73*l.* 5*s.* 7*d.*$\frac{1}{2}$.

111. If a gallon of brandy cost 1*l.* 3*s.* 6*d.* what cost 29$\frac{1}{4}$ gallons? *Ans.* 34*l.* 7*s.* 4*d.*$\frac{1}{2}$.

112. What will 21$\frac{3}{4}$ yards of lace cost, at 1*l.* 1*s.* 6*d.* per yard? *Ans.* 23*l.* 7*s.* 7*d.*$\frac{1}{2}$.

113. What will $37\frac{1}{2}$ lb. of nutmegs cost, at 1*l.* 4*s.* 8*d.* per lb.? *Ans.* 46*l.* 5*s.*

114. What will $87\frac{1}{4}$ gallons of oil cost, at 8*s.* 6*d.* per gallon? *Ans.* 37*l.* 5*s.* 10*d*$\frac{1}{2}$.

115. Required the value of a parcel, containing $567\frac{1}{4}$ hundred of Whitechapel needles, at 1*s.* 8*d.* per hundred. *Ans.* 47*l.* 5*s.* 5*d.*

PROPORTION.

126. [k] Proportion, called also the Golden Rule, and the Rule of Three, teaches from three numbers given (whereof two are of the same kind) to find a fourth: it consists of two branches, viz. The Rule of Three *Direct*, and The Rule of Three *Inverse*.

127. DIRECT PROPORTION, OR, THE RULE OF THREE DIRECT,

teaches from three numbers given to find a fourth, which (when the three numbers are properly arranged) will be as great when compared with the second, as the third is when compared with the first; so that, if the third be *greater* than the first, the fourth will also be *greater* than the second; and if the third be *less* than the first, the fourth will, in like manner, be *less* than the second.

128. [l] RULE I. Examine the question carefully, and when you

---

[k] The comparison of one number to another is called their *ratio;* and when of four given numbers the first has the same ratio to the second which the third has to the fourth, these four numbers are said to be *proportionals.*

Hence it appears, that ratio is the comparison of *two numbers*, but proportion is the equality of *two ratios:* we cannot then with propriety talk of the *proportion of one number to another*, nor confound the terms, as some authors have done. The name Proportion comes from the Latin *pro* and *portio*.

[l] The fundamental principle of the rules of Proportion is this, namely, If four numbers are proportionals, the product of the two extreme terms is equal to the product of the two means. Thus, since 2 is as great when compared with 3, as 4 is when compared with 6, 2 has the same ratio to 3 that 4 has to 6; and consequently these four numbers are proportionals, that is, 2 : 4 : : 3 : 6. Now the product of the extremes equals the product of the means, namely, $2\times6=4\times3$; and since these products are equal, we are at liberty to substitute one product for the other. And further, if any product be divided by one of its factors, the quotient will evidently be the other.

These particulars being premised, the rule will be easily accounted for as follows. Let the three terms 2 : 4 : : 3 be given to find the fourth: now 4 and 3

have discovered the three numbers or terms contained in it, you will find that two of them are of the same kind, and that the remaining term is of the same kind with the fourth term, or answer required. Also, of the two that are alike, *one* will be a term of supposition, and *the other* a term of demand.

II. State the question, that is, place the three given terms in a row from left to right; let the *odd* term (which is of the same kind with the answer) stand in the middle, and the two remaining terms (which are both of one kind) in the first and third places, observing to put the term of supposition in the first (or left hand) place, and the term of demand in the third; and place two dots vertically between the first and second terms, and four dots in form of a square between the second and third.

III. Reduce the first and third terms to the same denomination, (if they are not so already,) and if either or both of them consist of different denominations, both must be reduced to the lowest mentioned in either. Likewise the second term must be reduced to the lowest denomination mentioned in it.

IV. Multiply the second and third terms together, and divide the product by the first; the quotient will be the fourth term, or answer, in the same denomination into which the second term was reduced.

---

are the two means, and their product, viz. 4×3, equals the product of the extremes, and may be therefore taken for it: but we have one of the extremes, viz. 2, given; wherefore if the said product be divided by 2, the quotient will be the other extreme, that is, $\frac{4 \times 3}{2} = 6$, the other extreme or fourth term required: and the same may be shewn in every other case. Wherefore, in the Direct Rule of Proportion, if the second and third terms be multiplied together, and the product divided by the first term, the quotient will be the answer; which is the rule.

To make the rule perfectly clear, one or two more particulars will require an explanation. If four numbers are proportionals, that is, if the first be to the second as the third to the fourth, then will *the first be to the third as the second to the fourth;* and this accounts for the usual method of stating questions belonging to this rule: and the reason why the first and third terms must be reduced to the same denomination is, that numbers cannot be compared together except they are of the same denomination, as is evident; and hence it will readily appear, that the fourth term or answer will be in the same denomination with that to which the second was reduced.

V. The quotient will in most cases require to be reduced to a *higher* denomination; if there be a remainder, it will be of the same denomination which the second term was brought into, and must be reduced into the next *lower* denomination, and then divided by the first term, and the quotient will be of this latter denomination; and if there be still a remainder, it must be reduced lower and divided, and so on until you get to the lowest denomination the second term admits of.

Method of proof. Reverse the question thus; say as the third term is to the fourth, so is the first term to a fourth, or answer, which will come out exactly the same as the second term in the original question, when the work is right: or, to be more plain, make the third term the *first*, the answer the *second*, and the first term the *third*, and, working according to the rule, the answer will be the same as the second term of the question.

EXAMPLES.

1. If 2 yards 3 quarters of muslin cost 1*l.* 5*s.* 8*d.* what is the value of 4 yards 3 quarters 2 nails of the same?

OPERATION.

First term.		*Second term.*		*Third term.*
2*yd.* 3*qr.*	:	1*l.* 5*s.* 8*d.*	: :	4*yd.* 3*qr.* 2*n.*
4		20		4
11		25		19
4		12		4
44 *nails.*		308 *pence.*		78 *nails.*

The second and third terms multiplied together.

```
   308
    78
  ----
  2464
 2156
```

Divided by the 1*st.* 44)24024 (546 *quotient in pence.*

```
  220        12)
  ---      2|0)4|5 6
   202     2l. 5s. 6d. Ans. or 4th term.
   176
   ---
    264
    264
```

Proof.

First term, or 3d term of quest.		*Second term, or 4th of quest.*		*Third term, or 1st of quest,*
4*yd.* 3*qr.* 2*n.*	:	2*l.* 5*s.* 6*d.*	: :	2*yd.* 3*qr.*
4		20		4
19		45		11
4		12		4
78 *nails.*		546 *pence.*		44 *nails.*

The second and third terms multiplied together.

546
44
2184
2184

Divided by the 1*st.* 78)24024 (308 *quotient in pence.*
234 12)
624 2|0)2|5 8
624 1*l.* 5*s.* 8*d.* *Ans. or 4th term.*

Explanation.

Of the three terms given in the question, the odd term is money, viz. 1*l.* 5*s.* 8*d.* which (being of the same kind as the answer required) I put in the *second* place. The two remaining terms are of the same kind, viz. both muslin; these occupy the first and third places; the term of supposition, 4*yd.* 3*qr.* in the first, and the term of demand, 4*yd.* 3*qr.* 2*n.* in the third. I reduce both these terms into nails; the result of the first being 44, and of the third 78. I also reduce the second, 1*l.* 5*s.* 8*d.* into pence, and the result is 308. I then multiply 308 and 78, viz. the second and third, together, and divide the product 24024 by 44, the first. The quotient 546 is of the same name which the second term was in when I multiplied it by the third, namely, pence; I therefore divide 546 by 12 and 20 successively, and it gives 2*l.* 5*s.* 6*d.* the answer.

For the proof. I take the third term of the question and make it the *first*; the fourth term, or answer, I make the *second*; and the first term of the question I make the *third*; and, working exactly the same as before, the answer comes out 1*l.* 5*s.* 8*d.* which is the same as the second term of the question, and thereby shews that the operation is truly performed.

2. If 5*lb.* of powder-blue cost 10*s.* 5*d.* how many pounds can I buy for 2*l.* 16*s.* 3*d.*?

First term.		*Second term.*		*Third term.*
10*s.* 5*d.*	:	5*lb.*	: :	2*l.* 16*s.* 3*d.*
12				20
125 *pence.*				56
				12
				675 *pence.*
				5

125)3375(27*lb. Ans.*
250
875
875

3. If 3 ounces of tea cost 1s. 7d.$\frac{1}{2}$, what cost 1cwt. 1qr. 2lb.? *Ans.* 61l. 10s. 8d. *Stated thus;* 3oz. : 1s. 7d.$\frac{1}{2}$: : 1cwt. 1qr. 2lb.

4. If 11 pounds of sugar cost 13s. 6d.$\frac{1}{4}$, what cost 2cwt. 3qr. 4lb.? *Ans.* 19l. 3s. 6d.

Stated thus; 11lb. : 13s. 6d.$\frac{1}{4}$: : 2cwt. 3qr. 4lb.

5. If 7 yards of cloth cost 12l. 7s. 4d. what will 12yd. 2qr. 2n. cost? *Ans.* 22l. 6s. 1d.

Stated thus; 7yd. : 12l. 7s. 4d. : : 12yd. 2qr. 2n.

6. If 15 pounds of cheese cost 16s. 3d. what is that per cwt.? *Ans.* 6l. 1s. 4d. *Stated thus;* 15lb. : 16s. 3d. : : 112lb.

7. If hops sell for 3l. 18s. 6d. per cwt. what sum will buy 8cwt. 3qr. 14lb.? *Ans.* 34l. 16s. 8d.$\frac{1}{4}$.

Stating; 112lb. : 3l. 18s. 6d. : : 8cwt. 3qr. 14lb.

8. A Rear-Admiral's half-pay is 17s. 6d. per day, how much is that per year? *Ans.* 319l. 7s. 6d.

Stating; 1 *day* : 17s. 6d. : : 365 *days*.

9. A person owes 1234l. but being unable to pay the whole, his creditors agree to accept 16s. 6d.$\frac{1}{2}$ in the pound; what sum do they receive in all? *Ans* 1020l. 12s. 5d.

Stating; 1l. : 16s. 6d.$\frac{1}{2}$: : 1234l.

10. What sum will purchase 20 reams of paper, when 6s. 5d. will buy seven quires? *Ans.* 18l. 6s. 8d.

Stating; 7 *quires* : 6s. 5d. : : 20 *reams*.

11. A person having 3751 Venetian ducats, agrees to sell them at the rate of 40 ducats for 9l.; how much English money does he receive for the whole? *Ans.* 843l. 19s. 6d.

Stating; 40*duc.* : 9l. : 3751*duc.*

12. A bankrupt who owes 2500l. has effects amounting to 437l. 10s.; how much in the pound will he be able to pay his creditors? *Ans.* 3s. 6d. *Stating;* 2500l. : 437l. 10s. : : 1l.

13. If a gallon of wine cost 12s. 4d. what is the value of a pipe? *Ans.* 77l. 14s.

14. If 13 yards of cloth cost 2l. 0s. 1d. what must be given for 75 yards? *Ans.* 11l. 11s. 3d.

15. If a quarter of wheat cost 5l. 4s. 10d. what will 11qr. 3bu. cost? *Ans.* 59l. 12s. 5d.$\frac{3}{4}$.

16. What is the value of 30ch. 12bu. 2pk. of coals, at 6s. 6d. per sack? *Ans.* 118l. 7s. 1d.

129. *When the second term is a compound number,* it will in

many cases be convenient to use Compound Multiplication and Division, whereby the trouble of reducing both the second term and the answer will be saved.

17. If 3 cwt. of tallow cost 5l. 10s. 9d.$\frac{3}{4}$, what is the value of 16 cwt.?

By Comp. Mult. and Division.

cwt.		L.	s.	d.		cwt.
3	:	5	19	9$\frac{3}{4}$	::	16
				4		
		23	19	3		
				4		
	3)	95	17	0		
Ans.		31	19	0		

Explanation.

Here for 16 I multiply by 4 × 4, and then divide the product by 3, which gives the answer immediately, without any trouble in reducing.

By the common method.

cwt.		L.	s.	d.		cwt.
3	:	5	19	9$\frac{3}{4}$	::	16
		20				
		119				
		12				
		1437				
		4				
		5751				
		16				
		34506				
		5751				
	3)	92016(				
	4)	30672(				
	12)	7668(				
	2|0)	63|9(				
Ans.		31l.	19s.			

18. If 12 ounces of silver cost 3l. 7s. 6d. what cost 35 ounces? *Ans.* 9l. 16s. 10d.$\frac{1}{2}$.

19. If 15 loads of hay sell for 12l. 12s. 11d. what sum will be sufficient to purchase 28 loads at the same rate? *Ans.* 23l. 12s. 1d.$\frac{1}{4}$, 5 *rem.*

20. If 27 fother of lead cost 171l. 15s. 9d. what sum will buy 35 fother? *Ans.* 222l. 13s. 9d.

21. Sold 81 pounds of tobacco for 14l. 0s. 1d.$\frac{1}{2}$; what must be charged for 64 pounds? *Ans.* 11l. 1s. 4d.

22. Bought 99 reams of foolscap for 126l. 4s. 6d.; what will 137 reams cost at that rate? *Ans.* 174l. 13s. 6d.

130. *When the first term will divide the second without remainder.*

RULE. Divide the second term by the first, and multiply the quotient by the third; the product will be the answer.

23. Paid 4*l.* 17*s.* 6*d.* for the reaping of 10 acres of wheat; what must be paid for the reaping of $12\frac{3}{4}$ acres at that rate?

By the rule.

```
a.    L.   s.  d.      a.
10 :)4    17   6(: :  12¾
           9   9
              12¾
     5    17   0
           4  10½
           2   5¼
Ans. 6     4   3¾
```

Here I divide the second term by 10, and multiply the quotient by $12\frac{3}{4}$.

By the common method, Art. 128.

```
a.      L.   s.   d.       a.
10  :   4    17   6   : :  12¾
 4     20                   4
40     97                  51
       12
     1170
       51
     1170
    5850
 4|0)5967|0(3
  12)1491(3
  2|0)12|4(
Ans. 6l. 4s. 3d.¾.
```

By the preceding rule, Art. 129.

```
a.      L.   s.   d.       a.
10  :   4    17   6   : :  12¾
                 12¾
       58    10   0
        2     8   9
        1     4   4½
   10)62      3   1½
Ans.    6     4   3¾
```

When the terms are reduced by the common method, we may frequently work by this rule; in the present instance we cannot without reducing the second term into farthings, which may be done; but as no trouble is saved by it, the common method is preferable.

24. If 8 years' rent of a farm be 5232*l.* what sum will 11 years' rent amount to? *Ans.* 7194*l.*

25. If 9 bushels of wheat sell for 4*l.* 19*s.* 9*d.* what is that per load? *Ans.* 22*l.* 3*s.* 4*d.*

26. If 4*l.* be charged for a year's interest of 100*l.* what sum must be lent to gain 500*l.* in the same time? *Ans.* 12500.

27. If 10 ounces of gold be worth 40*l.* 15*s.* what will be charged for a cup of the same metal, weighing 2*lb.* 8*oz.*? *Ans.* 130*l.* 8*s.*

131. *When the first term will divide the third without remainder.*

RULE. Divide the third term by the first, and multiply the quotient by the second, or the second by the quotient; the product will be the answer.

28. If 6 yards of broad cloth cost 6*l.* 7*s.* 8*d.*$\frac{1}{2}$, what cost 54 yards?

> *Stating; 6yd. : 6l. 7s. 8d.*$\frac{1}{2}$ *: : 54yd. Here I divide* 54 *by* 6, *and the quotient is* 9. *I then multiply the second term by* 9, *and it gives* 57*l.* 9*s.* 4*d.*$\frac{1}{2}$ *for the answer.*

29. What is the value of 1 cwt. of sugar, when 4 lb. cost 4*s.* 9*d.*? *Ans.* 6*l.* 13*s.*

30. What must be given for a hundred dozen of eggs, at 1*s.* 2*d.*$\frac{1}{2}$ per score? *Ans.* 3*l.* 12*s.* 6*d.*

31. If 25 sheep eat up 4*a.* 3*r.* 11*p.* of turnips in a certain time, how many acres will serve a flock of ten thousand for the same time? *Ans.* 1927*a.* 2*r.*

32. What is the value of 1 cwt. of cheese, when 7 lb. cost 4*s.* 1*d.*$\frac{1}{2}$? *Ans.* 3*l.* 6*s.*

132. If the first term, or either the second or the third, (not both,) can be divided by any number without remainder, the quotients may be used instead of the terms from whence they arise.

33. If 21 yards of lace cost 35 shillings, what cost 29 yards?

> *Stating;* 21*yd.* : 35*s.* : : 29*yd. Divide the* 1*st and* 2*d by* 7, *and there will arise* 3 : 5 : : 29, *and working with these numbers instead of the numbers in the question, we obtain the answer* 2*l.* 8*s.* 4*d.*

34. If 6 horses eat 21 bushels of oats in a week, how many bushels will serve 30 horses the same time?

> *Stating;* 6*h.* : 21*bu.* : : 30*h. Here dividing the* 1*st and* 3*d by* 6, *we shall have* 1 : 21 : : 5, *whence we have the answer* 105 *bushels.*

35. Gave 7*s.* 6*d.* for 12 peaches; what will 29 cost at the same rate? *Ans.* 18*s.* 1*d.*$\frac{1}{2}$.

> *Stating:* 12*p.* : 7*s.* 6*d.* : : 29*p. The* 1*st and* 2*d will divide by* 12, *whence* 1*p.* : 7*d.*$\frac{1}{2}$: : 29*p.*

36. Bought 5 dozen of fowls for 4*l.* 14*s.* 6*d.*; what sum will a higler have to pay, who in the course of a year buys 100 score? *Ans.* 157*l.* 10*s.*

> *Stating;* 60*f.* : 4*l.* 14*s.* 6*d.* : : 2000*f. Divide the* 1*st and* 3*d by* 20, *and* 3*f.* : 4*l.* 14*s.* 6*d.* : : 100*f.*

PROMISCUOUS EXAMPLES FOR PRACTICE.

37. If a pound of tobacco cost 3*s*. 9*d*. what is that per cwt.? *Ans*. 21*l*.

38. Paid 1*l*. 19*s*. 10*d*.$\frac{1}{4}$ for 11 men's wages; what sum will pay 15 men for the same time? *Ans*. 2*l*. 14*s*. 4*d*.$\frac{1}{4}$.

39. Gave 16*l*. 1*s*. 9*d*. for reaping 33 acres of wheat; what will the reaping of a field of 5 acres cost me at that price? *Ans*. 2*l*. 8*s*. 9*d*.

40. Bought 1 cwt. of sugar for 3*l*. 14*s*. 8*d*. what is that per lb.? *Ans*. 8*d*.

41. If 32*EE*. 3*qr*. 1*n*. of cloth cost 8*l*. 3*s*. 3*d*. what is that a yard? *Ans*. 4*s*.

42. If the expence of housekeeping for 5 days be 1*l*. 17*s*. 9*d*. what is that per year? *Ans*. 137*l*. 15*s*. 9*d*.

43. What sum will purchase an ox, weighing 8*cwt*. 1*qr*. 23*lb*. at 3*s*. 8*d*. per stone? *Ans*. 21*l*. 14*s*. 0*d*.$\frac{1}{7}$.

44. What is the value of 19 quarters of wheat, at 24*l*. 6*s*. 8*d*. per load? *Ans*. 92*l*. 9*s*. 4*d*.

45. What will 7$\frac{1}{4}$ hundred of faggots cost, at 5*s*. 5*d*. a score? *Ans*. 10*l*. 3*s*. 1*d*.$\frac{1}{2}$.

46. How many hours will a person require to count 245000*l*. supposing he can count 250*l*. every 3 minutes? *Ans*. 49 *hours*.

47. At 3*s*. 6*d*. per week, how long can I keep my horse at grass for 5*l*.? *Ans*. 28 *weeks* 4 *days*.

48. If I rent an estate 735*l*. 10*s*. 6*d*. per annum, and pay 245*l*. 3*s*. 6*d*. poor's-rate, how much is that in the pound? *Ans*. 6*s*. 8*d*.

49. The week's wages of 9 Irish potatoe-diggers amount to 4 guineas; how many does a person employ who pays them weekly 106*l*. 17*s*. 4*d*. at the same rate? *Ans*. 229.

50. A foreigner on his arrival in England exchanges 3808 florins for 241*l*. 19*s*. 4*d*.; how much does he receive for each? *Ans*. 1*s*. 3*d*.$\frac{1}{4}$.

51. If ink be sold at 5*s*. 6*d*. per gallon, what must be given for a cask holding 12*gal*. 3*qt*. 1*pt*.? *Ans*. 3*l*. 10*s*. 9*d*.$\frac{3}{4}$.

52. If 13 horses eat up 10*a*. 2*r*. 10*p*. of grass, how many acres will be sufficient to supply the horses of a regiment 1200 strong for the same time? *Ans*. 975 *acres*.

53. What is the value of 5 fother of lead, at 16*s*. 4*d*. per cwt.? *Ans*. 79*l*. 12*s*. 6*d*.

55. If 8 lb. of bacon cost 7s. 4d. how many cwt. can be bought for 19l. 8s. 8d.? *Ans.* 3*cwt.* 3*qr.* 4*lb.*

55. If 4*cwt.* 3*qr.* 2*lb.* of cinnamon cost 139*l.* 8*s.* 6*d.* what sum will purchase 3*cwt.* 2*qr.* 4*lb.*? *Ans.* 103*l.* 7*s.* 10*d.*¼, 36 *rem.*

56. If 27*l.* 4*s.* 6*d.* buy 1*cwt.* 3*qr.* 2*lb.* of tea, what quantity can be bought for 89*l.* 14*s.* 3*d.*? *Ans.* 4*cwt.* 2*qr.* 3*lb.*

57. If rice be sold for 2*l.* 6*s.* 8*d.* per cwt., what quantity can be had for 52*l.* 12*s.* 1*d.*? *Ans.* 22*cwt.* 2*qr.* 5*lb.*

58. At 30*l.* 13*s.* 6*d.* per cent, what will be the gain on an adventure of 1309*l.* 17*s.* 10*d.*? *Ans.* 401*l.* 16*s.* 2*d.*, 5388 *rem.*

59. If 1*yd.* 2*qr.* 3*n.* of linen cost 1*l.* 2*s.* 3*d.* what cost 5*EE.* 4*qr.* 3*n.*? *Ans.* 4*l.* 18*s.* 0*d.*¼, 3 *rem.*

60. Paid 4*l.* 3*s.* 2*d.*½ for the carriage of 1*t.* 2*cwt.* 3*qr.* 4*lb.* what must be paid for the carriage of 4*t.* 3*cwt.* 2*qr.* 1*lb.* the same distance? *Ans.* 15*l.* 4*s.* 10*d.*¼, 561 *rem.*

61. Bought 102*a.* 3*r.* 30*p.* of land, giving at the rate of 248 guineas for 6 acres; what did the estate stand me in? *Ans.* 4467*l.* 9*s.* 9*d.*

62. What will a cargo of Peruvian bark, weighing 120*t.* 1*cwt.* 2*qr.* 3*lb.*, sell for, at 54*l.* 2*s.* 8*d.* per cwt.? *Ans.* 130002*l.* 13*s.*

134. INVERSE PROPORTION; OR, THE RULE OF THREE INVERSE,

teaches from three numbers given to find a fourth, such that whenever the third is *less* than the first, the fourth will be proportionally *greater* than the second; and when the third is greater than the first, the fourth will be always *less* than the second.

135. RULE I. State the question, and reduce the term as in the foregoing rule.

II. Multiply the first and second terms together, and divide the product by the third; the quotient will be the answer in the same denomination the second term was left in [m].

[m] It has been observed, that "Direct and Inverse Proportion are parts of the same general rule, and in a scientific arrangement it would be best to consider them in that manner." Thus, from the nature of any given question, it is easy to determine whether the fourth term ought to be greater or less than the second; if *greater*, the less extreme must be made the divisor, and the

Method of proof. Reverse the question as in the Rule of Three Direct.

EXAMPLES.

1. If 8 men can do a piece of work in 20 days, in what time will the same be accomplished by 16 men?

OPERATION.

m.		d.		m.
8	:	20	: :	16
		8		

16)160(10 *days Ans.*
16
0

Explanation.

Here, having stated the question, I multiply the first and second terms together, viz. 20 × 8, and divide the product 160 by 16 the third; the quotient 10 is the number of days required.

2. What quantity of calico, 5 quarters wide, will be sufficient to line a garment, containing 3*yd.* 3*qr.* of cloth, a yard and an half wide?

yd.	qr.		yd.	qr.		qr.
1	2	:	3	3	: :	5
4			4			
6			15			
			6			

5)90
4)18
Ans. 4*yd.* 2*qr.*

3. If 8 cwt. be carried 51 miles for a certain sum, how far ought 34 cwt. to be carried for the same money? *Ans.* 12 *miles.*

4. How many yards of oil cloth, 3 quarters wide, will be sufficient to cover the floor of a passage 2 yards and a quarter wide and 20 yards long? *Ans.* 60 *yards.*

5. Lent my friend 120*l.* for 5 months; what sum ought he to lend me for 9 months to requite my kindness? *Ans.* 66*l.* 13*s.* 4*d.*

---

other the multiplier; but if *less*, then the greater extreme must be made the divisor, and the *less* the multiplier.

The truth of the rule may be shewn from the first example; for if 8 men can do a piece of work in 20 days, 16 men will do *twice as much* in the same time, or *as much* in half the time, that is in 10 days, which is the answer: and the same may be shewn of any other example under the rule.

6. If when the quartern loaf cost 6*d.* the threepenny loaf weigh 2*lb.* 2*oz.* 12*dr.* what ought it to weigh when the quartern costs 8*d.*? *Ans.* 1*lb.* 10*oz.* 1*dr.*

7. If 2 cwt. be carried 204 miles for a certain sum, how much ought to be carried 24 miles for the same money? *Ans.* 17*cwt.*

8. How much must be cut off from a board 8 inches wide, to make a top to a stool 12 inches long and 12 wide? *Ans.* 18 *inc.*

9. How many crowns are equal in value to 200 half-guineas? *Ans.* 420.

10. A field of wheat can be reaped by 20 men in 4 days; now supposing only 8 men can be hired, how long will they require to reap the same? *Ans.* 10 *days.*

11. A person drinks 100 bottles of wine, at 2*s.* 6*d.* per bottle, in a year; now supposing the price of the wine increases to 4*s.* 3*d.* the bottle, how much may he drink without increasing the expence? *Ans.* 58 *bottles,* 42 *rem.*

12. A family takes 12 loaves, at 1*s.* 2*d.* per loaf, in a week, but bread rising, they make shift with 9 loaves, which cost exactly as much; what is the increased value of the loaf? *Ans.* 1*s.* 6*d.*$\frac{1}{7}$, 6 *rem.*

136. COMPOUND PROPORTION.

When five terms are given to find a sixth, this rule is called The Double Rule of Three, or The Rule of Five; also, whatever number of terms is given, it is usually called by the general name of Compound Proportion.

137. RULE I. Let that term be put in the second place which is of the same kind with the answer required.

II. Place the terms of supposition one above another in the first place, and the terms of demand one above another in the third, so that the first and third terms in each row may be of the same kind; let them be reduced to the same denomination, and the second to the lowest mentioned in it.

III. Examine each stating separately (using the second term in common for each) by saying, If the first term give the second, does the third term require more or less than the second? if more, mark the less extreme for a divisor; but if less, mark the greater extreme.

III. Multiply the unmarked numbers together for a dividend,

and the marked ones together for a divisor; divide the dividend by the divisor, and the quotient will be the answer in the same denomination the second term was brought into [m].

EXAMPLES.

1. If 6 men spend 154 shillings in 7 days, what sum will 8 men spend in 9 days?

Stating.

*6 *men*	:	154 *shil.*	: :	8 *men*
*7 *days*	:	———	: :	9 *days*

OPERATION.

$$\frac{154 \times 8 \times 9}{6 \times 7} = \frac{11088}{42} = 264 \textit{ shil.} = 13l.\ 4s.\ \textit{Answer.}$$

Explanation.

Here 154 shillings is of the same kind with the answer, and is therefore made the second term; 6 and 7 are evidently the terms of supposition, and therefore put in the first place; 8 and 9 being terms of demand are put in the third. I then say, If 6 men spend 154 shillings, 8 men will spend *more*; I therefore mark the *less* extreme 6 for a divisor: also, If 7 days spend 154 shillings, 9 days will require *more*; I therefore again mark the *less* extreme 7. Next I multiply the three unmarked numbers, viz. 154, 8, and 9, together for a dividend, and the product is 11088; and also the two marked numbers 6 and 7 for a divisor, and the product is 42. Dividing the former by the latter, the quotient is 264 shillings, which divided by 20 gives 13*l.* 4*s.* for the answer.

2. If 4 gallons of beer serve 5 persons 6 days, how many days will 7 gallons last 8 persons?

Stating.

*4 *gal.*	:	6 *days*	: :	7 *gal.*
5 *pers.*	:	———	: :	8 **pers.*

OPERATION.

$$\frac{6 \times 7 \times 5}{4 \times 8} = \frac{210}{32} = 6 \textit{ days } 13\tfrac{1}{2} \textit{ hours Ans.}$$

Explanation.

If 4 gallons serve 6 days, 7 gallons will serve *more*; I therefore mark the *less* 4. Again, if 5 persons are supplied 6 days, 8 persons will be supplied (with the same quantity) *less* than 6 days. I therefore mark the *greater* extreme 8, and proceed as before.

---

[m] The truth of this conclusion may be shewn by working the first example according to the rules of simple Proportion, namely, by employing two statings. Thus, 6 men : 154 shil. : : 8 men : $\frac{154 \times 8}{6}$ shil. Then 7 days : $\frac{154 \times 8}{6}$ shil. : : 9 days : $\frac{154 \times 8 \times 9}{6 \times 7}$ = 264 shil. = 13*l.* 4*s.* the answer as above. In the same manner every example in the rule may be proved; and it will furnish a profitable exercise for the industrious student to prove all his operations in this rule by two single rules of three statings.

3. If 100*l.* in 12 months gain 5*l.* interest, what sum will 80*l.* gain in 10 months?

Stating.

*100*l.*	:	5*l.*	: :	80*l.*
*12*mo.*	:	—	: :	10*mo.*

OPERATION.

$$\frac{5\times 80\times 10}{100\times 12}=\frac{4000}{1200}=3l.\ 6s.\ 8d.\ \textit{Answer.}$$

4. If 8 men in 9 weeks earn 75*l.* 12*s.* how many weeks must 11 men work to earn 100*l.*?

Here 75*l.* 12*s.* = 1512 *shil.* 100*l.*=2000 *shil.*

Stating.

8 *men*	:	9 *weeks*	: :	*11 *men*
*1512 *shil.*	:	———	: :	2000 *shil.*

OPERATION.

$$\frac{8\times 9\times 2000}{11\times 1512}=\frac{144000}{16632}=8\ \textit{weeks}\ 4\ \textit{days},\ 10080\ \textit{rem.}$$

5. If a carrier receive 2*l.* 2*s.* for the carriage of 3*cwt.* 150 miles, how much must be paid for the carriage of 7*cwt.* 3*qr.* 14*lb.* 100 miles? *Ans.* 3*l.* 13*s.* 6*d.*

6. If 10 acres of grass be mowed by 2 men in 7 days, how many acres can be mowed by 24 men in 14 days? *Ans.* 240 *acres.*

7. If a man earn 5 shillings a day, what sum will 64 men earn in 12½ days? *Ans.* 200*l.*

8. If 100*l.* in 12 months gain 3*l.* interest, in what time will 75*l.* gain 1*l.* 13*s.* 9*d.*? *Ans.* 9 *months.*

9. If 13*cwt.* be carried 20 miles for 2*l.* 10*s.* what weight can I have carried 50 miles for 3*l.* 3*s.* 6*d.*? *Ans.* 6*cwt.* 2*qr.* 11*lb.* 162 *rem.*

10. If 150*l.* gain 3*l.* 7*s.* 6*d.* in 9 months, what sum will gain 3*l.* in 12 months? *Ans.* 100*l.*

11. If 3 horses eat 12 bushels of oats in 16 days, how many quarters will 200 horses eat in 24 days? *Ans.* 150 *quarters.*

12. If 24 men can build a wall in 36 days, how many men would be required to do 5 times as much work in 3 days? *Ans.* 1440 *men.*

PRACTICE.

137. [a] Practice is an easy method of solving such rule of three questions as have unity for their first term. The operations consist of Compound Multiplication and Division; and the latter is performed by the given price, &c. taken in aliquot parts of an unit of some superior denomination.

138. One number is said to be an *aliquot part* of another, when the former is contained some number of times exactly in the other; that is, when the former will divide the latter without leaving any remainder.

Tables of Aliquot Parts.

1. Of a Penny.

$\frac{1}{2}$	$= \frac{1}{2}$
$\frac{1}{4}$	$= \frac{1}{4}$

2. Of a Shilling.

	s.
6*d.*	$= \frac{1}{2}$
4	$= \frac{1}{3}$
3	$= \frac{1}{4}$
2	$= \frac{1}{6}$
$1\frac{1}{2}$	$= \frac{1}{8}$
1	$= \frac{1}{12}$

3. Of a Pound.

s. d.	L.
10 0	$= \frac{1}{2}$
6 8	$= \frac{1}{3}$
5 0	$= \frac{1}{4}$
4 0	$= \frac{1}{5}$
3 4	$= \frac{1}{6}$
2 6	$= \frac{1}{8}$
2 0	$= \frac{1}{10}$
1 8	$= \frac{1}{12}$
1 0	$= \frac{1}{20}$

4. Of Sixpence.

3*d.*	$= \frac{1}{2}$
2	$= \frac{1}{3}$
$1\frac{1}{2}$	$= \frac{1}{4}$
1	$= \frac{1}{6}$
$\frac{3}{4}$	$= \frac{1}{8}$
$\frac{1}{2}$	$= \frac{1}{12}$

5. Of Threepence.

$1\frac{1}{2}d.$	$= \frac{1}{2}$
1	$= \frac{1}{3}$
$\frac{3}{4}$	$= \frac{1}{4}$
$\frac{1}{2}$	$= \frac{1}{6}$
$\frac{1}{4}$	$= \frac{1}{12}$

6. Of Fourpence.

2*d.*	$= \frac{1}{2}$
1	$= \frac{1}{4}$
$\frac{1}{2}$	$= \frac{1}{8}$

7. Of 10 Shillings.

s. d.	
5 0	$= \frac{1}{2}$
2 6	$= \frac{1}{4}$
2 0	$= \frac{1}{5}$
1 8	$= \frac{1}{6}$
1 3	$= \frac{1}{8}$
1 0	$= \frac{1}{10}$
6	$= \frac{1}{20}$

8. Of 5 Shillings.

s. d.	
2 6	$= \frac{1}{2}$
1 8	$= \frac{1}{3}$
1 3	$= \frac{1}{4}$
1 0	$= \frac{1}{5}$
10	$= \frac{1}{6}$
$7\frac{1}{2}$	$= \frac{1}{8}$
6	$= \frac{1}{10}$
5	$= \frac{1}{12}$

9. Of 4 Shillings.

s. d.	
2 0	$= \frac{1}{2}$
1 4	$= \frac{1}{3}$
1 0	$= \frac{1}{4}$
8	$= \frac{1}{6}$
d.	
6	$= \frac{1}{8}$
4	$= \frac{1}{12}$

10. Of 2*s.* and 6*d.*

s. d.	
1 3	$= \frac{1}{2}$
10	$= \frac{1}{3}$
$7\frac{1}{2}$	$= \frac{1}{4}$
6	$= \frac{1}{5}$
5	$= \frac{1}{6}$
$3\frac{3}{4}$	$= \frac{1}{8}$
3	$= \frac{1}{10}$
$2\frac{1}{2}$	$= \frac{1}{12}$

11. Of 2 Shillings.

s. d.	
1 0	$= \frac{1}{2}$
8	$= \frac{1}{3}$
6	$= \frac{1}{4}$
4	$= \frac{1}{6}$
3	$= \frac{1}{8}$
2	$= \frac{1}{12}$

[a] The rules of Practice are particularly useful to merchants and traders, in computing the value of their commodities. These rules are likewise useful in the Mathematics; and it is on account of their ready and convenient application to the computations, which occur in almost every branch of science, that they are introduced in this place.

139. *Preparatory direction.* Write down the given number, namely, the number which expresses the quantity of goods of which you want to find the value, and to the left of it draw three perpendicular lines sufficiently wide apart to contain a column of figures in each of the two intervals; then, having (if necessary) broken the given price into sums which are respectively aliquot parts of some greater whole, and of one another, you are to place these sums in their proper order under one another in the left hand interval; and opposite each, in the right hand interval, place the number expressing what aliquot part it is; then, work according to the directions given in the rule to which the particular question belongs.

140. *When the price is an aliquot part of a penny.*

RULE I. Having drawn the lines, and placed the price and the aliquot part as above directed, divide the given number by the aliquot part, and the quotient will be the answer in pence; these (if there be a sufficient number) must be divided by 12 and 20, to reduce them into pounds.

II. If when you have divided by the aliquot part there be a remainder, it will be pence, and must be reduced to farthings, which being divided by the aliquot part, will give farthings*.

EXAMPLES.

1. What is the value of 3571 yards of tape, at $\frac{1}{2}d.$ per yard?

OPERATION.

$\frac{1}{2}$	$\frac{1}{2}$	3571
	12	1785 $\frac{1}{2}$
	2|0	14|8 9

Answer 7*l.* 8*s.* 9*d.* $\frac{1}{2}$

Explanation.

Here the given price (a halfpenny) is *one half* of a penny; I put the halfpenny in the left hand column, and the half in the right; I then divide by 2, and the quotient is 1785, with 1 remaining; this 1 penny I make 4 farthings, which divided by 2 gives $\frac{1}{2}$; I then divide successively by 12 and 20.

* Nothing can be easier to understand than the rules of Practice. In the rule here given let us take Example 1; where, if 3571 yards had cost a penny each, they would evidently have amounted to 3571 pence; but as they cost only $\frac{1}{2}d.$ each, it is plain that they will amount to half that number of pence. And in Example 2, 9867 pears at $\frac{1}{4}d.$ each will, it is plain, amount to one quarter that number of pence: wherefore this rule is evident.

2. What cost 9867 pears, at $\frac{1}{4}d.$ each.

OPERATION.

$\frac{1}{4}$	$\frac{1}{4}$	9867
	12	2466 $\frac{3}{4}$
	2|0	20|5 6

Answer 10*l.* 5*s.* 6*d.*$\frac{3}{4}$

Explanation.

A farthing is *one fourth* of a penny; I therefore divide by 4; the 3 remainder are pence, which reduced are 12 farthings, which 12 divided by 4 give $\frac{3}{4}$. I then divide by 12 and 20 as before.

3. 4968 at $\frac{1}{4}$. *Answer* 5*l.* 3*s.* 6*d.*
4. 2807 at $\frac{1}{2}$. *Answer* 5*l.* 16*s.* 11*d.*$\frac{1}{2}$.
5. 7013 at $\frac{1}{4}$. *Answer* 7*l.* 6*s.* 1*d.*$\frac{1}{4}$.
6. 3465 at $\frac{1}{2}$. *Answer* 7*l.* 4*s.* 4*d.*$\frac{1}{2}$.

141. *When the price is an aliquot part of a shilling.*

RULE. Having placed the price, and its corresponding aliquot part, as before directed, divide by the aliquot part, and the quotient will be shillings, which reduce to pounds by dividing it by 20. If there be a remainder after the first division, it is shillings; reduce it to pence, which divide by the aliquot part, and the quotient is pence [p].

7. What is the worth of 826 lemons, at 3*d.* each?

OPERATION.

d.		
3	$\frac{1}{4}$	826
	2|0	20|6 6

Answer 10*l.* 6*s.* 6*d.*

Explanation.

Here 3*d.* by table 2, is $\frac{1}{4}$ of a shilling; I therefore divide by 4; the quotient is shillings, and the remainder 2 I make 24 pence, which divided by 4 gives 6 pence; the quotient I then divide by 20, which gives the answer.

---

[p] Here if we consider the articles as worth a shilling each, they will cost in the whole as many shillings as there are articles given; wherefore if they cost an aliquot part of a shilling each, the whole will evidently amount to the same part of so many shillings: thus, in Example 7, 826 lemons at 1*s.* each will cost 826 shillings; but at 3*d.* each (since 3*d.* is $\frac{1}{4}$ of a shilling) they will amount to $\frac{1}{4}$ of 826 shillings: and the like in other cases.

8.

d.
1 | $\frac{1}{12}$ | 583 at 1d.
2|0 | 4|8 7
Answer 2l. 8s. 7d.

9.

d.
$1\frac{1}{2}$ | $\frac{1}{8}$ | 829 at $1\frac{1}{2}$d.
2|0 | 10|3 $7\frac{1}{2}$
Answer 5l. 3s. 7d.$\frac{1}{2}$

10.

d.
6 | $\frac{1}{2}$ | 732 at 6d.
2|0 | 36|6
Answer 18l. 6s.

11. 514 at 1d. *Answer* 2l. 2s. 10d.
12. 603 at 1d.$\frac{1}{2}$. *Answer* 3l. 15s. 4d.$\frac{1}{2}$.
13. 345 at 2d. *Answer* 2l. 17s. 6d.
14. 472 at 3d. *Answer* 5l. 18s.
15. 123 at 4d. *Answer* 2l. 1s.
16. 207 at 6d. *Answer* 5l. 3s. 6d.

142. *When the price is an aliquot part of a pound.*

RULE I. Having placed the price and the aliquot part as before, divide by the aliquot part, and the quotient will be pounds.

II. The remainder (if any) must be reduced to shillings, and divided by the aliquot part for shillings; if there be a second remainder reduce it to pence, and divide for pence; if a third, reduce it to farthings, and divide for farthings[q].

17. What must be given for 135 pullets, at 2s. 6d. each?

OPERATION.

s. d.
2 6 | $\frac{1}{8}$ | 135
Answer 16l. 17s. 6d.

Explanation.

Here by table 3, 2s. 6d. is $\frac{1}{8}$ of a pound; I divide by 8, and the quotient is 16 pounds; the remainder 7 turned into shillings is 140, which divided by 8 gives 17 shillings, with 4 remainder; the latter brought into pence is 48, which divided by 8 gives 6 pence.

---

[q] The observations in the two foregoing notes are equally applicable to this rule; thus, in Example 17, 135 pullets at 1l. each will cost 135l.; but since the price (2s. 6d.) is $\frac{1}{8}$ of a pound, it is evident that the whole will amount to $\frac{1}{8}$ of 135l.: and the like may be shewn of the other Examples.

18.

s. d.		
3 4	$\frac{1}{6}$	217 at 3s. 4d.
	Answer	36l. 3s. 4d.

19.

s.		
5	$\frac{1}{4}$	123 at 5s.
	Answer	30l. 15s.

20.

s. d.		
6 8	$\frac{1}{3}$	307 at 6s. 8d.
	Answer	102l. 6s. 8d.

21. 716 at 1s. *Answer* 35l. 16s.
22. 624 at 1s. 8d. *Answer* 52l.
23. 513 at 2s. *Answer* 51l. 6s.
24. 308 at 2s. 6d. *Answer* 38l. 10s.
25. 213 at 3s. 4d. *Answer* 35l. 10s.
26. 701 at 4s. *Answer* 140l. 4s.
27. 405 at 5s. *Answer* 101l. 5s.
28. 109 at 10s. *Answer* 54l. 10s.

143. *When the price is not an aliquot part.*

RULE I. Divide the price into sums which are aliquot parts of the whole, (viz. of a shilling, or a pound, as the case may be,) or of which one is an aliquot part of the whole, and the rest aliquot parts either of the whole, of this, or successively of one another.

II. Divide by these several aliquot parts, and the quotients being added together, the sum will be the answer in shillings, if the aliquot parts are of a shilling; and in pounds, if they are aliquot parts of a pound[r].

[r] This rule will be evident from an explanation of the 29th Example, where if 371oz. had cost 1s. each, the whole would have cost 371 shillings; but at 3d. each (since 3d. is $\frac{1}{4}$ of a shilling) they will cost $\frac{1}{4}$ of 371 shillings, or 92s. 9d.; and at $\frac{1}{2}$d. each (since $\frac{1}{2}$d. is $\frac{1}{6}$ of 3d.) they will cost $\frac{1}{6}$ of their value at 3d., that is $\frac{1}{6}$ of 92s. 9d., or 15s. 5d.$\frac{1}{2}$; wherefore, if the value at 3d. each be added to the value at $\frac{1}{2}$d. each, the sum will evidently be the value at 3d.$\frac{1}{2}$ each: and the like may be shewn in every other case.

29. What is the value of 371 ounces of tobacco, at $3d.\frac{1}{2}$ per ounce*?

Operation.

d.		
3	$\frac{1}{4}$	371
$\frac{1}{2}$	$\frac{1}{6}$	92 9
		15 $5\frac{1}{2}$
	2|0	10|8 $2\frac{1}{2}$

Answer $5l.\ 8s.\ 2d.\frac{1}{2}$

Explanation.

Here $3d.\frac{1}{2}$ not being an aliquot part of a shilling, I divide it into two sums, $3d.$ and $\frac{1}{2}$, of which $3d.$ is $\frac{1}{4}$ of a shilling, and $\frac{1}{2}d.$ $\frac{1}{6}$ of $3d.$; I divide the given number 371 by 4, and the quotient is $92s.\ 9d.$; this, which is the value at $3d.$, I divide by 6, because $\frac{1}{2}d.$ is $\frac{1}{6}$ of $3d.$; and the quotient is $15s.\ 5d.\frac{1}{2}$; I add the two quotients together, and reduce the 108 shillings into pounds, by dividing by 20.

Or thus;

d.		
2	$\frac{1}{6}$	371
$1\frac{1}{2}$	$\frac{1}{8}$	61 10
		46 $4\frac{1}{2}$
	2|0	10|8 $2\frac{1}{2}$

Answer $5l.\ 8s.\ 2d.\frac{1}{2}$

In the second operation, I divide the given price $3d.\frac{1}{2}$ differently; I take $2d.$, which is $\frac{1}{6}$, and $1d.\frac{1}{2}$, which is $\frac{1}{8}$, both aliquot parts of a shilling; I therefore divide *the top line by both*, add the quotients together, and divide by 20, which gives the answer the same as in the first operation.

30. What sum will 123 sawyers earn, at $17s.\ 6d.$ each†?

Operation.

s.		
10	$\frac{1}{2}$	123
5	$\frac{1}{2}$	61 10
2 6	$\frac{1}{2}$	30 15
		15 7 6

Answer $107l.\ 12s.\ 6d.$

Explanation.

I find that $17s.\ 6d.$ will conveniently resolve into aliquot parts, viz. $10s. = \frac{1}{2}$ of a pound, $5s. = \frac{1}{2}$ of $10s.$, and $2s.\ 6d. = \frac{1}{2}$ of $5s.$ I therefore divide successively by these, and the sum of the quotients is the answer.

31. What sum will pay for 215 cheeses, at $17s.\ 10d.$ each?

Operation.

s.		
10	$\frac{1}{2}$	215
5	$\frac{1}{2}$	107 10
2	$\frac{1}{5}$	53 15
10d	$\frac{1}{6}$	21 10
		8 19 2

Answer $191l.\ 14s.\ 2d.$

Explanation.

When there are several aliquot parts, we are not obliged to take each part out of the *next* preceding one; we may take it out of any of the preceding parts, as may be most convenient. In this example $2s.$ is $\frac{1}{5}$ (not of $5s.$ but) of $10s.$; I therefore divide $107l.\ 10s.$ (and not $53l.\ 15s.$) by it. In like manner $10d.$ is $\frac{1}{6}$ (not of $2s.$ but) of $5s.$; I therefore divide $53l.\ 15s.$ (and not $21l.\ 10s.$) by 6.

* To prove this example, multiply 371 by $3d.\frac{1}{2}$, and divide by 12 and 20.

† To prove operations of this kind, multiply the given price by the number of particulars (Art. 105.); in the present instance it will be $17s.\ 6d. \times 123$.

32. What cost 423 cwt. of iron plates, at 18*s*. 7*d*.½ per cwt.?

1st method.

s.				
10	$\frac{1}{2}$	423		
4	$\frac{1}{5}$	211	10	
4	$\frac{1}{5}$	84	12	
6*d*.	$\frac{1}{8}$	84	12	
$1\frac{1}{2}$	$\frac{1}{4}$	10	11	6
		2	12	$10\frac{1}{2}$
Answer		393	18	$4\frac{1}{2}$

2d method.

s.				
10	$\frac{1}{2}$	423		
5	$\frac{1}{2}$	211	10	
2 6	$\frac{1}{2}$	105	15	
1	$\frac{1}{5}$	52	17	6
$1\frac{1}{2}$	$\frac{1}{8}$	21	3	
		2	12	$10\frac{1}{2}$
Answer		393	18	$4\frac{1}{2}$

3d method.

s.				
10	$\frac{1}{2}$	423		
5	$\frac{1}{2}$	211	10	
2	$\frac{1}{5}$	105	15	
1	$\frac{1}{2}$	42	6	
6*d*.	$\frac{1}{2}$	21	3	
$1\frac{1}{2}$	$\frac{1}{4}$	10	11	6
		2	12	$10\frac{1}{2}$
Answer		393	18	$4\frac{1}{2}$

Explanation.

In the 1st, the first three aliquot parts are taken out of the top line, and the remaining two in succession. In the 2d, the first three parts are taken in succession, the fourth is taken out of the second, and the fifth out of the fourth. In the 3d, the third aliquot part is taken out of the first, and all the rest in succession.

33.

d.				
6	$\frac{1}{2}$	438 at 10*d*.¼		
3	$\frac{1}{2}$	219		
1	$\frac{1}{3}$	109	6	
$\frac{1}{4}$	$\frac{1}{4}$	36	6	
		9	$1\frac{1}{2}$	
	2|0	37|4	$1\frac{1}{2}$	
	Ans.	18*l*. 14*s*. 1*d*.½		

34.

d.				
6	$\frac{1}{2}$	716 at 11*d*.¾		
4	$\frac{1}{3}$	358		
1	$\frac{1}{4}$	238	8	
$\frac{3}{4}$	$\frac{1}{8}$	52	8	
		44	9	
	2|0	70|1	1	
	Ans.	35*l*. 1*s*. 1*d*.		

35.

s.				
5	$\frac{1}{4}$	803 at 7*s*. 11*d*.½		
2	$\frac{1}{10}$	200	15	
10*d*	$\frac{1}{6}$	80	6	
1	$\frac{1}{10}$	33	9	2
$\frac{1}{2}$	$\frac{1}{2}$	3	6	11
		1	13	$5\frac{1}{2}$
Answer		319	10	$6\frac{1}{2}$

36. 341 at 1*d*.¼ *Ans.* 1*l*. 15*s*. 6*d*.¼

37. 489 at 1*d*.¾ *Answer* 3*l*. 11*s*. 3*d*.¾

38. 780 at 5*d*.½ *Answer* 17*l*. 17*s*. 6*d*.

39. 619 at 7*d.*$\frac{3}{4}$ *Answer* 19*l.* 19*s.* 9*d.*$\frac{1}{4}$
40. 215 at 9*d.*$\frac{3}{4}$ *Answer* 8*l.* 14*s.* 8*d.*$\frac{1}{4}$
41. 703 at 10*d.*$\frac{1}{4}$ *Answer* 30*l.* 0*s.* 5*d.*$\frac{3}{4}$
42. 407 at 11*d.* *Answer* 18*l.* 13*s.* 1*d.*
43. 806 at 11*d.*$\frac{1}{4}$ *Answer* 37*l.* 15*s.* 7*d.*$\frac{1}{2}$
44. 721 at 1*s.* 7*d.* *Answer* 57*l.* 1*s.* 7*d.*
45. 364 at 2*s.* 5*d.* *Answer* 43*l.* 19*s.* 8*d.*
46. 205 at 5*s.* 11*d.* *Answer* 60*l.* 12*s.* 11*d.*
47. 268 at 8*s.* 3*d.*$\frac{1}{4}$ *Answer* 110*l.* 16*s.* 7*d.*
48. 410 at 9*s.* 7*d.*$\frac{3}{4}$ *Answer* 197*l.* 14*s.* 9*d.*$\frac{1}{2}$
49. 104 at 10*s.* 5*d.*$\frac{1}{2}$ *Answer* 54*l.* 7*s.* 8*d.*
50. 153 at 16*s.* 4*d.*$\frac{3}{4}$ *Answer* 125*l.* 8*s.* 6*d.*$\frac{3}{4}$
51. 816 at 18*s.* 2*d.*$\frac{1}{4}$ *Answer* 742*l.* 1*s.*
52. 194 at 19*s.* 8*d.* *Answer* 190*l.* 15*s.* 4*d.*
53. 121 at 19*s.* 11*d.*$\frac{3}{4}$ *Answer* 120*l.* 17*s.* 5*d.*$\frac{3}{4}$

144. *When the price is greater than one shilling, but less than two shillings.*

RULE I. Let the given number stand for the value at a shilling each, consequently no line must be drawn under it.

II. Take the pence and farthings in aliquot parts of a shilling, and of each other, divide by them as before, add the quotients and the given number together, and divide by 20 [x].

54. What is the value of 572 lb. of soap, at 1*s.* 3*d.*$\frac{1}{2}$ per lb.?

OPERATION.

d.			
3	$\frac{1}{4}$	579	
$\frac{1}{2}$	$\frac{1}{6}$	144	9
		24	$1\frac{1}{2}$
	2|0	74|7	$10\frac{1}{2}$
Answer		37*l.* 7*s.*	10*d.*$\frac{1}{2}$

Explanation.

I draw no line under the given number 579, but let it stand for the value of the soap at 1*s.* per pound. I then say, 3*d.* is $\frac{1}{4}$ of a shilling, and $\frac{1}{2}$ is $\frac{1}{6}$ of 3*d.*; and having divided by them as in the former examples, I add the quotients and the given number 579 together, and divide the shillings in the sum by 20 for the answer.

[x] This rule will be easily understood from the preceding notes. Thus, example 54, 579 lb. at 1*s.* each will amount to 579 shillings, and at 3*d.* each (since 3*d.* is $\frac{1}{4}$ of 1*s.*) to $\frac{1}{4}$ of 579 shillings, or 144*s.* 9*d.*, and at $\frac{1}{2}$ each (since $\frac{1}{2}$ is $\frac{1}{6}$ of 3*d.*) to $\frac{1}{6}$ of the price at 3*d.*, or $\frac{1}{6}$ of 144*s.* 9*d.* or 24*s.* 1*d.*$\frac{1}{2}$. These three values (viz. the value at 1*s.* each, the value at 3*d.* each, and the value at $\frac{1}{2}$ each) will, being added together, evidently give the value at 1*s.* 3*d.*$\frac{1}{2}$ each.

55.

d.		
6	$\frac{1}{2}$	376 at 1s. 7d.$\frac{3}{4}$
$1\frac{1}{2}$	$\frac{1}{4}$	188
$\frac{1}{4}$	$\frac{1}{6}$	47
		7 10
	2|0	61|8 10

Answer 30*l.* 18*s.* 10*d.*

56.

d.		
6	$\frac{1}{2}$	413 at 1s. 11d.$\frac{1}{2}$
4	$\frac{1}{3}$	206 6
$1\frac{1}{2}$	$\frac{1}{4}$	137 8
		51 $7\frac{1}{2}$
	2|0	80|8 $9\frac{1}{2}$

Answer 40*l.* 8*s.* 9*d.*$\frac{1}{2}$

57.

d.		
6	$\frac{1}{2}$	265 at 1s. 11d.$\frac{3}{4}$
3	$\frac{1}{2}$	132 6
2	$\frac{1}{3}$	66 3
$\frac{3}{4}$	$\frac{1}{4}$	44 2
		16 $6\frac{3}{4}$
	2|0	52|4 $5\frac{1}{4}$

Answer 26*l.* 4*s.* 5*d.*$\frac{1}{4}$

58. 287 at 1*s.* 1*d.* *Answer* 15*l.* 10*s.* 11*d.*
59. 371 at 1*s.* 2*d.*$\frac{1}{2}$ *Answer* 22*l.* 8*s.* 3*d.*$\frac{1}{2}$
60. 247 at 1*s.* 4*d.*$\frac{3}{4}$ *Answer* 17*l.* 4*s.* 9*d.*$\frac{1}{4}$
61. 494 at 1*s.* 5*d.*$\frac{1}{4}$ *Answer* 35*l.* 10*s.* 1*d.*$\frac{1}{2}$
62. 627 at 1*s.* 6*d.* *Answer* 47*l.* 0*s.* 6*d.*
63. 251 at 1*s.* 8*d.*$\frac{3}{4}$ *Answer* 21*l.* 14*s.* 0*d.*$\frac{1}{4}$
64. 526 at 1*s.* 9*d.*$\frac{1}{2}$ *Answer* 47*l.* 2*s.* 5*d.*
65. 625 at 1*s.* 11*d.* *Answer* 59*l.* 17*s.* 11*d.*

145. *When the price is an even number of shillings.*

RULE. Multiply the given number by half the number of shillings; double the product of the right hand figure, call this shillings, and the rest of the product will be pounds[y].

[y] If the given number be multiplied by the price of each, the product will be the value of the given number in the same denomination with the price, as is evident. Wherefore, ex. 66, if 328 be multiplied by 4, the product is 1312 shillings, which divided by 20 is 65*l.* 12*s.*, the answer. Now multiplying by 4 and dividing by 20, is equivalent to multiplying by 2 and dividing by 10; the product then by this last method is 65 and 6 over; this 6 is 6 tenths; and therefore to reduce it to shillings (or twentieths of a pound) the 6 must evidently be doubled: wherefore the rule is plain.

66. What is the value of 328 four-shilling stamps?

OPERATION.

```
    328
      2
Ans. 65l. 12s.
```

Explanation.

Here I multiply by 2 (the half of 4), and the product of the first figure is 16, which doubled is 32 shillings; or 1*l.* 12*s.* I put down 12, and carry 1; then twice 32 are 64 and 1 carried 65, for pounds.

```
        67.             68.             69.
     347 at 2s.      534 at 4s.      659 at 6s.
        1               2               3
Answer 34l. 14s.    106l. 16s.      197l. 14s.
```

70. 262 at 8*s.* *Answer* 104*l.* 16*s.*

71. 456 at 10*s.* *Answer* 228*l.*

72. 481 at 12*s.* *Answer* 228*l.* 12*s.*

73. 753 at 14*s.* *Answer* 527*l.* 2*s.*

74. 500 at 16*s.* *Answer* 400*l.*

75. 819 at 18*s.* *Answer* 737*l.* 2*s.*

146. *When the price is any number of shillings.*

RULE. Multiply the given number by the price, and divide the product by 20[a].

76. What is the value of 123 fowls, at 3 shillings each?

OPERATION.

```
     123
       3
 2|0)36|9
Ans. 18l. 9s.
```

Explanation.

Here I multiply 123 by 3, and divide the product 369 by 20, and the quotient is the answer.

```
        77.             78.             79.
     543 at 7s.      728 at 9s.      613 at 11s.
        7               9               11
  2|0)380|1       2|0)655|2       2|0)674|3
Ans. 190l. 1s.     327l. 12s.      337l. 3s.
```

80. 912 at 5*s.* *Answer* 228*l.*

81. 962 at 12*s.* *Answer* 577*l.* 4*s.*

82. 215 at 15*s.* *Answer* 161*l.* 5*s.*

83. 215 at 16*s.* *Answer* 172*l.*

84. 794 at 17*s.* *Answer* 674*l.* 18*s.*

85. 742 at 19*s.* *Answer* 704*l.* 18*s.*

---

[a] The reason of this rule will be sufficiently plain, from what has been said in the preceding note.

147. *When the price is shillings and pence, or shillings, pence, and farthings.*

RULE I. Multiply by the shillings, take the pence and farthings in aliquot parts, and divide by them as before, add the several quotients and the product together, and divide the shillings by 20.

Or, II. Take the whole price in aliquot parts of a pound and of one another; divide by them, and add the quotients together for the answer*.

86. Required the value of 213 stone of beef, at 7*s.* 8*d.*$\frac{3}{4}$ per stone.

OPERATIONS.

By Rule 1.

6*d.*	$\frac{1}{2}$	213		
		7		
		1491		
2	$\frac{1}{3}$	106	6	
$\frac{3}{4}$	$\frac{1}{8}$	35	6	
		14	$3\frac{3}{4}$	
	2|0	164|6	$3\frac{3}{4}$	

Answer 82*l.* 6*s.* 3*d.*$\frac{3}{4}$

By Rule 2.

5*s.*	$\frac{1}{4}$	213		
2	$\frac{1}{10}$	53	5	
6*d.*	$\frac{1}{4}$	21	6	
2	$\frac{1}{3}$	5	6	6
$\frac{3}{4}$	$\frac{1}{8}$	1	15	6
			13	$3\frac{3}{4}$

Answer 82 6 $3\frac{3}{4}$

Explanations.

In the first place I multiply 213 by the 7 shillings, and the product is the value at 7*s.* Next I say, 6*d.* is the $\frac{1}{2}$ of a shilling, and divide the top line by 2, the quotient is the value at 6*d.* Next I say, 2*d.* is $\frac{1}{3}$ of 6*d.*; I divide 106..6 by 3, and the quotient is the value at 2*d.* In like manner $\frac{3}{4}$ is $\frac{1}{8}$ of 6*d.*; I therefore divide 106..6 by 8, and the quotient is the value at $\frac{3}{4}$. These added together, and the shillings divided by 20, we have the answer.

In the second operation, I say, 5*s.* is $\frac{1}{4}$ of a pound, and 2*s.* is $\frac{1}{10}$ of a pound; I divide the top line by both these, and the quotient is pounds. I then say, 6*d.* is $\frac{1}{4}$ of 2*s.*; 2*d.* $\frac{1}{3}$ of 6*d.*; and $\frac{3}{4}$ $\frac{1}{8}$ of 6*d.* I divide 5..6..6 by the two latter, then add all the quotients together, which gives the answer the very same as in the former operation.

* If the given number be multiplied by the shillings in the price of each, the product will be the price of the whole at so many shillings each; and the taking the odd pence and farthings in aliquot parts of a shilling and of each other has been already accounted for. If in the second rule we take the second operation of Ex. 86, and suppose 213 articles at 1*l.* each, they will amount to 213*l.*; wherefore at 5*s.* (or $\frac{1}{4}$ of a pound) they will amount to $\frac{1}{4}$ of 213 pounds; at 2*s.* each (or $\frac{1}{10}$ of a pound) they will amount to $\frac{1}{10}$ of 213 pounds; at 6*d.* each (or $\frac{1}{4}$ of 2*s.*) they will amount to $\frac{1}{4}$ the value at 2*s.* each; in like manner at 2*d.* each they will amount to $\frac{1}{3}$, and at $\frac{3}{4}$ to $\frac{1}{8}$ the value at 6*d.* each.

87.

6d.	$\frac{1}{2}$	765 at 5s. 9d.$\frac{3}{4}$
		5
		3825
3	$\frac{1}{2}$	382 6
$\frac{3}{4}$	$\frac{1}{4}$	191 3
		47 9$\frac{3}{4}$
	2|0	444|6 6$\frac{3}{4}$

Answer 222*l.* 6*s.* 6*d.*$\frac{3}{4}$

88.

6d.	$\frac{1}{2}$	293 at 12s. 10d.$\frac{1}{2}$
4	$\frac{1}{3}$	12
		3516
		146 6
$\frac{1}{2}$	$\frac{1}{8}$	97 8
		12 2$\frac{1}{2}$
	2|0	377|2 4$\frac{1}{2}$

Answer 188*l.* 12*s.* 4*d.*$\frac{1}{2}$

89.

10s.	$\frac{1}{2}$	136 at 16s. 8d.$\frac{1}{2}$		
5	$\frac{1}{2}$	68		
1	$\frac{1}{5}$	34		
6d.	$\frac{1}{2}$	6	16	
2	$\frac{1}{3}$	3	8	
$\frac{1}{2}$	$\frac{1}{4}$	1	2	8
			5	8
Answer		113	12	4

90. 704 at 3*s.* 6*d.* *Answer* 123*l.* 4*s.*
91. 493 at 6*s.* 4*d.* *Answer* 156*l.* 2*s.* 4*d.*
92. 512 at 7*s.* 6*d.* *Answer* 192*l.*
93. 701 at 15*s.* 4*d.*$\frac{1}{2}$ *Answer* 538*l.* 17*s.* 10*d.*$\frac{1}{2}$.
94. 894 at 17*s.* 6*d.*$\frac{3}{4}$ *Answer* 785*l.* 0*s.* 10*d.*$\frac{1}{2}$.
95. 251 at 14*s.* 7*d.*$\frac{1}{4}$ *Answer* 183*l* 5*s.* 7*d.*$\frac{3}{4}$.
96. 123 at 18*s.* 9*d.* *Answer* 116*l.* 6*s.* 3*d.*
97. 271 at 19*s.* 2*d.*$\frac{1}{2}$ *Answer* 260*l.* 5*s.* 5*d.*$\frac{1}{2}$.

148. *When the price is more than one pound, and less than two pounds.*

RULE I. Multiply by the number of shillings contained in the price, take parts for the pence and farthings, and proceed as in the last rule.

Or, II. Let the top line stand for the value at one pound, draw no line under it, and take parts for the shillings, pence, and farthings, as in the latter part of the preceding rule[b].

[b] This rule will be sufficiently clear from the preceding notes.

98. What sum will purchase 237 baskets of figs, at 1*l.* 6*s.* 9*d.*½ per basket?

OPERATIONS.

By Rule 1.

6*d.*	½	237		
		26		
		1422		
		474		
		6162		
3	½	118	6	
½	1/6	59	3	
		9	10½	
	2|0	634|9	7½	
Answer		317*l.*	9*s.*	7*d.*½

Explanations.

In the first operation I multiply 237 by 26, the number of shillings in 1*l.* 6*s.*; then I take parts for the remainder of the price, (viz. 9*d.*½,) 6*d.* ½ of a shilling, 3*d.* ½ of 6*d.*, and ½*d.* 1/6 of 3*d.*; I divide by these in order, and add up the quotients together with the first mentioned product; then I divide by 20 to reduce the shillings to pounds.

In the second operation, I let the given number 237 represent the value at 1 pound; I then take the rest of the price in aliquot parts, namely, 5*s.* ¼ of a pound, 1*s.* 1/5 of 5*s.*, 6*d.* ½ of 1*s.*, 3*d.* ½ of 6*d.*, and ½*d.* 1/6 of 3*d.*, and having divided by these, I add the quotients and top line together for the answer.

By Rule 2.

5*s.*	¼	237		
1	1/5	59	5	
6*d.*	½	11	17	
3	½	5	18	6
½	1/6	2	19	3
			9	10½
Answer		317	9	7½

99.

6*d.*	½	516 at 1*l.* 13*s.* 6*d.*¾.		
		33		
		1548		
		1548		
		17028		
¾	1/8	258		
		32	3	
	2|0	1731|8	3	
Answer		865*l.*	18*s.*	3*d.*

100.

10*s.*	½	428 at 1*l.* 16*s.* 6*d.*		
5	½	214		
1	1/5	107		
6*d.*	½	21	8	
		10	14	
Answer		781	2	0

101. 135 at 1*l.* 2*s.* 6*d.* *Answer* 151*l.* 17*s.* 6*d.*

102. 153 at 1*l.* 8*s.* 9*d.* *Answer* 219*l.* 18*s.* 9*d.*

103. 137 at 1*l.* 17*s.* 6*d.*¼. *Answer* 257*l.* 0*s.* 4*d.*¼.

104. 160 at 1*l.* 18*s.* 8*d.* *Answer* 309*l.* 6*s.* 8*d.*

149. ***When the price is pounds, shillings, pence, and farthings.***

RULE. Multiply the given number by the pounds, take parts for the rest of the price, and proceed as in the former rules [b].

105. What will 571 cwt. of sugar cost, at 2*l.* 12*s.* 9*d.*½ per cwt.?

OPERATION.

s.				
10	½	571		
		2		
		1142		
2	⅕	285	10	
6*d.*	¼	57	2	
3	½	14	5	6
½	⅙	7	2	9
		1	3	9½
Answer		1507	4	0½

Explanation.

Here I multiply the given number by 2, the number of pounds; I then take parts for the 12*s.* 9*d.*½, divide, and add all the quotients and the product by 2 together, and the result is the answer.

106.

s.				
4	⅕	237 at 12*l.* 4*s.* 8*d.*		
		12		
d.		2844		
8	⅙	47	8	
		7	18	
Answer		2899	6	

107.

s.				
10	½	359 at 25*l.* 10*s.* 10*d.*		
		25		
		1795		
d.		718		
10	$\frac{1}{12}$	179	10	
		14	19	2
Answer		9169	9	2

108. 643 at 4*l.* 6*s.* 8*d.* *Answer* 2786*l.* 6*s.* 8*d.*
109. 719 at 3*l.* 15*s.* 2*d.*¼. *Answer* 2702*l.* 19*s.* 9*d.*¾.
110. 298 at 18*l.* 8*s.* 0*d.* *Answer* 5483*l.* 4*s.* 0*d.*
111. 243 at 21*l.* 16*s.* 6*d.* *Answer* 5303*l.* 9*s.* 6*d.*
112. 121 at 30*l.* 19*s.* 9*d.*½. *Answer* 3749*l.* 14*s.* 9*d.*½.

150. *When the given number consists of a whole number and parts.*

RULE. Work for the whole number according to the directions given in the former rules, and add in such a part of the price as the question requires [c].

---

[b] If the given number be multiplied by the pounds each costs, the product will evidently be the value for the pounds; the reasons on which the method of taking the aliquot parts for the remainder of the price is founded, have been already explained.

[c] The price of ½ will evidently be (half the price of 1, or) half the given

113. What is the value of 109½ dozen of wine, at 2*l.* 6*s.* 6*d.* per dozen?

5*s.*	⅛	109½			
		2			
		218			*value at 2l. per dozen.*
1	⅕	27	5		*at 5s.*
6*d.*	½	5	9		*at 1s.*
		2	14	6	*at 6d.*
		1	3	3	= ½ *of 2l. 6s. 6d. or the value of ½.*
Answer		254	11	9	

114. 237¾ yards of cambric, at 12*s.* 4*d.* per yard?

4*d.*	⅓	237¾			
		12			
		2844			*value at 12s. per yard.*
		79			*at 4d.*
		6	2		*of ½ yard* } = ¾ *yard.*
		3	1		*of ¼ yard* }
		2|0)293|2	3		
Answer		146*l.* 12*s.* 3*d.*			

115. 142¼ at 2*l.* 10*s.* 6*d.* *Answer* 359*l.* 16*s.* 3*d.*
116. 273¼ at 7*s.* 6*d.* *Answer* 102*l.* 9*s.* 4*d.*½.
117. 234¾ at 17*s.* *Answer* 199*l.* 10*s.* 9*d.*
118. 120¾ at 5*l.* 15*s.* 5*d.* *Answer* 696*l.* 16*s.* 6*d.*¼.

151. *When the given number is of several denominations.*

RULE. Multiply the given price by the highest denomination of the given number, and take the remaining denominations of the given number in aliquot parts of the highest, and of one another, and work as in the preceding rules[d].

---

price; in like manner the price of any part will evidently be the same part of the given price: the rule is therefore manifest.

[d] The reason of the rule will appear from an examination of the 119th Example; where, since 1 cwt. costs 8*l.* 12*s.* 4*d.* it is plain that 15 cwt. will cost 15 times 8*l.* 12*s.* 4*d.* (or 129*l.* 5*s.*); 1 quarter will cost ¼ of 8*l.* 12*s.* 4*d.* (or 2*l.* 3*s.* 1*d.*), and 14 lb. will cost half what a quarter costs, (or 1*l.* 1*s.* 6*d.*½); and that these several quotients being added together, the sum will be the answer.

The Examples under this rule should be proved by The Rule of Three.

119. *What will 15*cwt.* 1*qr.* 14*lb.* of tobacco cost, at 8*l.* 12*s.* 4*d.* per cwt.?

OPERATION.

qr.		*L.*	*s.*	*d.*	
1	$\frac{1}{4}$	8	12	4	*the value of* 1*cwt.*
				5	
		43	1	8	
				3	
lb.		129	5	0	*the value of* 15*cwt.*
14	$\frac{1}{2}$	2	3	1	*of* 1 *quarter.*
		1	1	$6\frac{1}{2}$	*of* 14*lb.*
Answer		132	9	$7\frac{1}{2}$	*the value of* 15*cwt.* 1*qr.* 14*lb.*

Explanation.

Having put down the given price, I multiply it by 15, that is, by 5×3, the number of cwts.; I then say, 1 qr. is $\frac{1}{4}$ of a cwt., and 14 lb. $\frac{1}{2}$ of a qr.; I divide the top line by 4, and the quotient 2*l.* 3*s.* 1*d.* by 2, and add the whole together for the answer.

The proof is stated thus, 112*lb.* : 8*l.* 12*s.* 4*d.* : : 15*cwt.* 1*qr.* 14*lb.*

---

* The following tables of aliquot parts will be necessary in solving these Examples.

Of an cwt.	
2 qr. or 56 lb.	= $\frac{1}{2}$
1 qr. or 28 lb.	= $\frac{1}{4}$
16 lb.	= $\frac{1}{7}$
14 lb.	= $\frac{1}{8}$
8 lb.	= $\frac{1}{14}$
7 lb.	= $\frac{1}{16}$

Of a quarter.	
14 lb.	= $\frac{1}{2}$
7 lb.	= $\frac{1}{4}$
4 lb.	= $\frac{1}{7}$
$3\frac{1}{2}$ lb.	= $\frac{1}{8}$
2 lb.	= $\frac{1}{14}$

Of an acre.	
2 roods	= $\frac{1}{2}$
1 rood	= $\frac{1}{4}$

Of a rood.	
20 poles	= $\frac{1}{2}$
10 poles	= $\frac{1}{4}$
8 poles	= $\frac{1}{5}$
5 poles	= $\frac{1}{8}$
4 poles	= $\frac{1}{10}$
2 poles	= $\frac{1}{20}$

Of a hhd. of wine.	
9 gallons	= $\frac{1}{7}$
7 gallons	= $\frac{1}{9}$
3 gallons	= $\frac{1}{21}$

Of a gall. of wine.	
2 quarts	= $\frac{1}{2}$
1 quart	= $\frac{1}{4}$
1 pint	= $\frac{1}{8}$

To find the aliquot parts of any number, divide it successively by 2, 3, 4, &c. to half the given number, and reserve all the quotients which arise without remainders, then each of these quotients, and its divisor, will be aliquot parts one by the other; thus in 1 cwt. or 112 lb. I find that 14 will go 8 times, therefore 14 lb. is $\frac{1}{8}$, and 8 lb. $\frac{1}{14}$ of an hundred weight.

120.

17*cwt.* 3*qr.* 21*lb.* at 1*l.* 2*s.* 3*d.* per cwt.

qr.		*L.*	*s.*	*d.*	
2	½	1	2	3	*Here* 4×4+1=17.
				4	
		4	9	0	
				4	
		17	16	0	*value of* 16 *cwt.*
		1	2	3	1 *cwt.*
1	½		11	1½	2 *qrs.*
14*lb.*	½		5	6¾	1 *qr.*
7	½		2	9¼	14 *lb.*
			1	4½	7 *lb.*
Answer		19	19	1	*value of* 17*cwt.* 3*qr.* 21*lb.*

121.

134 *acres,* 3 *roods,* 16 *poles,* at 2*l.* 12*s.* 6*d.* per acre.

r.		*L.*	*s.*	*d.*	
2	½	2	12	6×4	*Here* 10×10+30+4=134.
				10	
		26	5	0×3	
				10	
		262	10	0	*value of* 100 *acres.*
		78	15	0	30 *acres.*
		10	10	0	4 *acres.*
1	½	1	6	3	2 *roods.*
10*p.*	¼		13	1½	1 *rood.*
5	½		3	3¼	10 *poles.*
1	⅕		1	7½	5 *poles.*
				3¾	1 *pole.*
Answer		353	19	7	*value of* 134*a.* 3*r.* 16*p.*

122. What is the value of 12*cwt.* 1*qr.* 14*lb.* of rice, at 2*l.* 12*s.* 6*d.* per cwt.? *Ans.* 32*l.* 9*s.* 8*d.*¼.

123. What will 25*cwt.* 2*qr.* 14*lb.* of tobacco cost, at 11*l.* 12*s.* 6*d.* per cwt.? *Ans.* 297*l.* 17*s.* 9*d.*¾.

124. What will 78*cwt.* 3*qr.* 12*lb.* of currants cost, at 5*l.* 15*s.* 6*d.* per cwt.? *Ans.* 455*l.* 8*s.*

125. Required the value of 23*cwt.* 3*qr.* 8*lb.* of soap, at 3*l.* 19*s.* 11*d.* per cwt. *Ans.* 95*l.* 3*s.* 8*d.*½.

126. Find the value of 17*cwt.* 1*qr.* 16*lb.* of iron, at 1*l.* 10*s.* 4*d.* per cwt. *Ans.* 26*l.* 7*s.* 7*d.*

127. What must be paid for 59*cwt.* 2*qr.* 24*lb.* of salt, at 2*l.* 17*s.* 4*d.* per cwt.? *Ans.* 171*l.* 3*s.* 7*d.*$\frac{1}{4}$.

128. At 2*l.* 9*s.* 6*d.* per cwt. what must be given for 22*cwt.* 3*qr.* 21*lb.* of currants? *Ans.* 56*l.* 15*s.* 4*d.*$\frac{3}{4}$.

129. What is the yearly rent of 145 *acres*, 1 *rood*, 32 *poles* of land, at 20*l.* 10*s.* 6*d.* per acre? *Ans.* 2985*l.* 7*s.* 2*d.*$\frac{1}{4}$.

130. What will 1234*hdd.* 28*gal.* 3*qt.* of port wine cost, at 60*l.* per hhd.? *Ans.* 74067*l.* 7*s.* 7*d.*$\frac{1}{4}$.

FRACTIONS.

152. A fraction is a number which denotes one or more parts of unity; the number 1 being supposed divisible into any number of equal parts at pleasure: whatever expresses any assigned number of those parts is called a fraction.

Fractions are usually divided into Vulgar, Decimal, Duodecimal, and Sexagesimal[f].

VULGAR FRACTIONS.

153. Vulgar Fractions are those which express the parts of unity, into whatever number of equal parts it may be supposed to be divided.

154. A Vulgar Fraction is denoted by two numbers placed one over the other, with a small line between them, thus, $\frac{4}{7}$.

155. The number below the line is called the *denominator*; it shews how many parts the unit is supposed to be divided into.

156. The number above the line is called the *numerator*; it shews how many of the aforesaid parts are to be understood by the fraction.

Thus in the above fraction $\frac{4}{7}$, the number 7 is the denominator, and shews that 1 is supposed to be divided into 7 equal parts; 4 is the numerator, and shews that 4 of those parts are to be understood by the fraction; that is, the value of the fraction is *four sevenths*, or four of such equal parts of which seven just make the number 1.

[f] The word *fraction* is derived from the Latin *frango*, to break, and is a name descriptive of the numbers included under it. Decimals and Duodecimals will be distinctly treated of; Sexagesimals (or Sixtieths) will likewise be explained when we treat of Practical Geometry, Trigonometry, &c. The term *vulgar* comes from *vulgus*, the common people. Vulgar Fractions mean fractions that admit of any denomination whatever.

157. When the numerator is *less* than the denominator, it is evident that the fraction expresses *fewer* parts than 1 is supposed to be divided into; consequently the value of the fraction is less than 1: such a fraction is called a *proper fraction;* thus, $\frac{1}{2}$ (one half), $\frac{2}{3}$ (two thirds), $\frac{3}{11}$ (three elevenths), &c. are proper fractions.

158. When the numerator is *equal to* the denominator, the fraction expresses just so many parts as 1 is supposed to be divided into; consequently its value will be equal to 1. In like manner, when the numerator is *greater* than the denominator, the fraction expresses *more* parts than 1 is divided into, and its value is *greater* than 1. In either case the fraction is called an *improper fraction;* thus, $\frac{4}{4}$ (four fourths), $\frac{5}{5}$ (five fifths), $\frac{7}{3}$ (seven thirds), $\frac{21}{9}$ (twenty-one ninths), &c. are improper fractions.

159. But this division is not confined to unity; we may conceive the parts themselves susceptible of a similar division; every fraction may be subdivided into other parts or fractions, and these into others, and so on without end: the expression denoting a fraction arising from such a division and subdivision of unity is called a *compound fraction;* thus, $\frac{1}{2}$ of $\frac{1}{2}$ (one half of one half), $\frac{2}{3}$ of $\frac{3}{5}$ (two thirds of three fifths), $\frac{1}{4}$ of $\frac{2}{7}$ of $\frac{3}{8}$ (one fourth of two sevenths of three eighths); such expressions as these consisting of two or more fractions, with the word *of* interposed between them, are, as is evident, meant to express a part of a part, or parts of parts, and are called compound fractions.

160. A simple fraction is that which is expressed by one numerator and one denominator, and both whole numbers.

161. When a whole number and a fraction are connected, so that both together form but one number, such an expression is called a *mixed number;* thus, $1\frac{2}{7}$ (one and two sevenths), $5\frac{8}{9}$ (five and eight ninths), $14\frac{1}{6}$ (fourteen and one sixth), &c. are mixed numbers.

162. When either the numerator or the denominator is a mixed number, or when both are mixed, the fraction is called a *complex fraction;* thus, $\frac{3\frac{1}{2}}{4}$ (three and one half, fourths), $\frac{2}{3\frac{5}{6}}$ (two, three and five sixths), $\frac{2\frac{2}{3}}{3\frac{4}{5}}$ (two and two thirds, three and four fifths[g]), &c. are complex fractions.

---

[g] When both terms are complex, this reading will scarcely be intelligible; perhaps it will be better to read these fractions in the following manner; viz. $\frac{3\frac{1}{2}}{4}$ three and one half *by* four; $\frac{2}{3\frac{5}{6}}$ two *by* three and five sixths; $\frac{2\frac{2}{3}}{3\frac{4}{5}}$ two and two thirds *by* three and four fifths; and the like of others.

163. A *common measure* of two or more numbers is such a number as will divide each without leaving any remainder; and the *greatest common measure* is the *greatest* number possible that will divide each without remainder.

164. A *common multiple* of two or more numbers is a number which each of them will divide without remainder; and the *least common multiple* is the least number that each of them will so divide.

REDUCTION OF VULGAR FRACTIONS.

165. Reduction of Fractions is the changing them from one form to another without altering their values. By the operations of reduction, fractions are expressed in the most convenient form for the readily understanding of their values, and likewise prepared for adding, subtracting, multiplying, and dividing[b].

166. *To find the greatest common measure of two numbers.*

RULE I. Divide the greater number by the less, and divide the divisor by the remainder.

II. Proceed in this manner, dividing continually the last divisor by the last remainder, until nothing remains: the last di-

---

There are some other denominations of fractions, as *continued fractions*, used for approximating to indeterminate ratios in small numbers; *vanishing fractions*, the properties of which are best explained by Fluxions, &c.

[b] Previous to entering on the Reduction of Fractions it will be proper to remark, that one fraction is always equal to another when the numerator of the first is to its denominator as the numerator of the second is to its denominator; thus, $\frac{1}{2}$ is equal to $\frac{2}{4}$ or to $\frac{3}{6}$ or to $\frac{4}{8}$ or to $\frac{10}{20}$ or to $\frac{50}{100}$ or to $\frac{100}{200}$, &c. for

$$1 : 2 :: \left\{\begin{array}{rcl} 2 & : & 4 \\ 3 & : & 6 \\ 4 & : & 8 \\ 10 & : & 20 \\ 50 & : & 100 \\ 100 & : & 200, \text{\&c.} \end{array}\right.$$

Hence it follows, that since numbers have the same ratio to one another that their like multiples or like parts have respectively, both terms of any fraction may be either multiplied or divided by any (the same) number without altering the original value of the fraction; thus both terms of $\frac{20}{30}$ may be multiplied by 7, 9, &c. or divided by 10, 5, &c. and the results $\frac{140}{210}$, $\frac{180}{270}$, $\frac{2}{3}$, $\frac{4}{6}$, &c. will be equal to each other and to the given fraction $\frac{20}{30}$.

visor is the greatest common measure of the two given numbers as was required [i].

EXAMPLES.

1. Required the greatest common measure of 72 and 120.

OPERATION.

```
72)120(1
   72
   --
   48)72(1
      48
      --
      24)48(2
         48
         --
```

Explanation.

First I divide the greater by the less, viz. 120 by 72, and the remainder is 48. Next I divide 72 by 48, and the remainder is 24. Lastly, I divide 48 by 24, and there is no remainder, wherefore the last divisor 24 is the greatest common measure required.

Ans. 24 *the greatest common measure.*

2. Required the greatest common measure of 536 and 792.

```
536)792(1
    536
    ---
    256)536(2
        512
        ---
         24)256(10
            24
            --
            16)24(1
               16
               --
                8)16(2
                  16
                  --
```

Answer 8 *the common measure required.*

3. What is the greatest common measure of 12 and 15? *Ans.* 3.

---

[i] If any number measures two other numbers, it will measure both their sum and difference; this being premised, the truth of the rule may be shewn from the first Example; thus, 24 measures 48, as appears by there being no remainder; wherefore it measures 48 + 24, or 72; and since it measures both 48 and 72, it likewise must measure 48 + 72, or 120; wherefore 24 is a common measure of 72 and 120.

It is likewise the greatest common measure; for if not, let there if possible be a greater; then, since this greater measures 72 and 120, it will measure their difference, (viz. 120 — 72 or) 48; and since it measures 72 and 48, it will likewise measure their difference, (or 72 — 48) = 24; wherefore a number greater than 24 will measure 24, which is absurd; wherefore 24 is the greatest common measure.

4. What is the greatest common measure of 376 and 940? *Ans.* 188.

5. What is the greatest common measure of 144 and 240? *Ans.* 48.

6. What is the greatest common measure of 1376 and 9402? *Ans.* 2.

167. *To find the greatest common measure of three or more numbers.*

RULE I. Find the greatest common measure of any two of the given numbers by the last rule.

II. Find the greatest common measure of this common measure and another of the given numbers by the last rule.

III. Find the greatest common measure of this last common measure and another of the given numbers, and so on until all the given numbers have been taken; the last common measure is the greatest common measure of all the given numbers [k].

7. Required the greatest common measure of 32, 48, and 68.

OPERATION.

```
First 32)48(1
         32
         --
         16)32(2
            32
            --

Then 16)68(4
        64
        --
         4)16(4
           16
           --
   Ans. 4.
```

Explanation.

First I find the greatest common measure of 32 and 48, (by Art. 166,) which is 16.

Then I find the greatest common measure of 16 and the remaining number 68, which is 4; therefore 4 is the greatest common measure of the three given numbers 32, 48, and 68, as was required.

---

[k] Having found the greatest common measure of two of the given numbers, if this measures the third, it will evidently be the greatest common measure of all the three; but if not, then it is equally evident that the greatest common measure of the said common measure and of the third number will be the greatest common measure of the three given numbers; and in the same manner the greatest common measure of four or more numbers may be accounted for.

8. What is the greatest common measure of 144, 216, and 324?

The greatest common measure of 144 *and* 216, *by Art.* 166, *is* 72, *and the greatest common measure of* 72 *and* 324 *is* 36, *the greatest common measure required.*

9. What is the greatest common measure of 88, 132, 154, and 165?

The greatest com. meas. of 88 *and* 132 *is* 44.
of 44 *and* 154 *is* 22.
of 22 *and* 165 *is* 11, *the answer req.*

10. What is the greatest common measure of 72, 120, and 132? *Ans.* 12.

11. What is the greatest common measure of 376, 940, 1034, and 1081? *Ans.* 47.

12. Required the greatest common measure of 100, 200, 350, 425, and 505. *Ans.* 5.

168. *To find the least common multiple of two given numbers.*

RULE I. Find the greatest common measure of the two given numbers, by Art. 166.

II. Multiply the two given numbers together, and divide the product by the greatest common measure; the quotient is the least common multiple required[1].

13. What is the least common multiple of 12 and 18?

The greatest common measure of 12 *and* 18, *found by Art.* 166, *is* 6; *thus,*

12)18(1
   12
   ——
    6)12(2
      12
      ——

And their product $12 \times 18 = 216$.

Therefore $\frac{216}{6} = 36$ *the least common multiple required.*

[1] For the greatest common measure of two numbers will evidently measure their product; and if a number greater than the greatest common measure be assumed, then one of the two numbers being divided by it, the quotient will be a fraction, and the other given number being multiplied by this fraction, the product will not be a multiple of that number; whence the rule is plain.

14. **What is the least common multiple of 48 and 72?**

Their greatest common measure (by Art. 166.) *is* 24, *and their product* $48 \times 72 = 3456$.

$$\textit{Therefore } \frac{3456}{24} = 144 \textit{ the answer.}$$

15. **To find the least common multiple of 30 and 40.**

Greatest common measure 10.

$$\textit{Therefore } \frac{30 \times 40}{10} = \frac{1200}{10} = 120 \textit{ the answer.}$$

16. **What is the least common multiple of 15 and 20?** *Ans.* **60.**

17. **Find the least common multiple of 60 and 84.** *Ans.* **420.**

18. **Required the least common multiple of 108 and 162.** *Ans.* **324.**

169. *To find the least common multiple of three or more numbers*[m].

Rule I. Find the least common multiple of any two of the numbers by the last rule.

II. Find the least common multiple of this multiple and another of the numbers, and it will be the answer for *three* numbers.

III. Find the least common multiple of this last multiple and another of the numbers, and it will be the answer for *four* numbers.

IV. Proceed in this manner until you have obtained the least common multiple of all the given numbers.

---

[m] The rule given by Mr. Bonnycastle is as follows:

"1. Divide by any number that will divide two or more of the given numbers without remainder, and set the quotients, together with the undivided numbers, in a line beneath."

"2. Divide the second line as before, and so on till there are no two numbers that can be divided; then the continued product of the divisors, quotients," and undivided numbers, "will give the multiple required."

Thus, to find the least common multiple of 4, 10, and 15.

$$\begin{array}{r|rrr} 5 & 4 & 10 & 15 \\ \hline 2 & 4 & 2 & 3 \\ \hline 2 & 2 & 1 & 3 \\ \hline & 1 & 1 & 3 \end{array}$$

Therefore $5 \times 2 \times 2 \times 3 = 60$ *the least common multiple required.*

19. What is the least common multiple of 4, 10, and 15?

The least common multiple of 4 and 10 (by Art. 168) is $\frac{4 \times 10}{2} = \frac{40}{2} = 20$.

The least com. mult. of 20 and 15 is $\frac{20 \times 15}{5} = \frac{300}{5} = 60$.

Therefore 60 is the least common multiple required.

20. What is the least common multiple of 12, 16, and 30?

Least common multiple of 12 and 16 is $\frac{12 \times 16}{4} = 48$.

Least common multiple of 48 and 30 is $\frac{48 \times 30}{6} = 240$ *Ans.*

21. Find the least common multiple of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

Here, because 8 is a multiple of 1, 2, and 4, every multiple of 8 will be some multiple of 1, 2, and 4; in like manner 6 being a multiple of 3, every multiple of 6 will be some multiple of 3; wherefore in the operation the numbers 1, 2, 3, and 4, may be omitted, as being aliquot parts of some of the other numbers. To find the least common multiple therefore of all the nine digits, we have only to find that of 5, 6, 7, 8, and 9.

The least common multiple of 5 and 6 is $\frac{5 \times 6}{1} = 30$; *of 30 and 7 is* $\frac{30 \times 7}{1} = 210$; *of 210 and 8 is* $\frac{210 \times 8}{2} = 840$; *and of 840 and 9 is* $\frac{840 \times 9}{3} = 2520$ *the answer.*

22. What is the least common multiple of 3, 4, and 8? *Ans.* 24.

23. What is the least common multiple of 3, 5, 8, and 10? *Ans.* 120.

24. What is the least common multiple of 3, 8, 14, and 22? *Ans.* 1848.

170. *To reduce a fraction to its lowest terms.*

RULE I. Find the greatest common measure of the numerator and denominator by Art. 166.

II. Divide both terms of the fraction by the greatest common measure, and the quotients will be the numerator and denomi-

nator respectively of the fraction which expresses the lowest terms of the given fraction[n].

25. Reduce $\frac{144}{186}$ to its lowest terms.

First, find the greatest common measure by Art. 166, *thus,*

$$\begin{array}{r} 144)186(1 \\ \underline{144} \\ 42)144(3 \\ \underline{126} \\ 18)42(2 \\ \underline{36} \\ \textit{greatest common measure } 6)18(3 \\ \underline{18} \end{array}$$

Secondly, divide both terms of the given fraction by the greatest common measure 6, *thus;* $6)\frac{144}{186} = \frac{24}{31}$ *the answer.*

26. Reduce $\frac{36}{48}$ to its lowest terms.

The greatest common measure by Art. 166 *is* 12.

Therefore $12)\frac{36}{48} = \frac{3}{4}$ *the lowest terms required.*

27. Reduce $\frac{648}{816}$ to its lowest terms. *Ans.* $\frac{27}{34}$.

Common measure 24.

28. Reduce $\frac{144}{168}$ to its lowest terms. *Ans.* $\frac{6}{7}$.

---

[n] Two fractions are equal to each other when the numerator of one has to its denominator the same ratio which the numerator of the other has to its denominator, as is plain from the definition of a fraction; hence it follows that the same fraction may be expressed in a great variety of ways; thus, $\frac{1}{2}$ is the same as $\frac{2}{4}$ or $\frac{3}{6}$ or $\frac{4}{8}$ or $\frac{5}{10}$ or $\frac{12}{24}$ or $\frac{21}{42}$, &c. &c.; this being premised, the above rule teaches to find the least numbers possible that will express any given fraction; if a fraction be not in its lowest terms, both terms must evidently be *divided*, and they must both be divided by *the same number*, otherwise the terms would not be proportionals, and therefore the resulting fraction would not equal the given one; (see note on Art. 165.): moreover this divisor must be the *greatest possible*, (viz. the greatest common measure of both terms,) otherwise the quotients will not be the least possible: whence the rule is plain.

29. Reduce $\frac{30}{125}$ to its lowest terms. *Ans.* $\frac{6}{25}$.

30. Reduce $\frac{2592}{3456}$ to its least terms. *Ans.* $\frac{3}{4}$.

31. Reduce $\frac{2064}{3432}$ to the lowest terms possible. *Ans.* $\frac{86}{143}$.

171. *To reduce a fraction to its lowest terms, without first finding the greatest common measure.*

RULE I. Divide both terms of the fraction by any number which will divide both without remainder, and the quotients will form another fraction equal to the former, but in lower terms.

II. Divide both terms of this latter fraction by any number that will exactly divide them, and the quotients will form another fraction in still lower terms than the last, and equal to it.

III. Proceed in this manner as long as division can be made, and the last fraction will be the lowest terms required*.

---

* Any number that will divide both terms of a fraction will evidently produce an equal fraction *in lower terms*; now if this last fraction be in like manner reduced, and likewise the resulting one, and so on continually as long as division can be made, it is plain the last fraction arising from this continued operation will be the given fraction in its least terms. To assist the operations in this rule it may be remarked, that,

1. Any number ending with an even number or a cipher is divisible by 2.

2. If the right hand place be a cipher, the number is divisible by 10.

3. Any number ending with 5 or 0 is divisible by 5.

4. If the two right hand figures be divisible by 4, the number is divisible by 4.

5. If the three right hand figures be divisible by 8, the number is divisible by 8.

6. If the sum of the digits constituting any number be divisible by 3 or 9, the whole is divisible by 3 or 9.

7. If the right hand digit be even, and the sum of all the digits divisible by 6, the whole will be divisible by 6.

8. If the sum of the first, third, fifth, digits be equal to the sum of the second, fourth, sixth, &c. the number is divisible by 11.

9. A number which is not a square, nor divisible by some number less than its square root, is a prime.

10. All prime numbers, except 2 and 4, have either 1, 3, 7, or 9, in the units place: all other numbers are composite.

11. When numbers having the sign + of addition, or — of subtraction,

32. Reduce $\frac{720}{864}$ to its lowest terms.

OPERATION.

$$\overset{6)}{\frac{720}{864}}=\overset{4)}{\frac{120}{144}}=\overset{3)}{\frac{30}{36}}=\overset{2)}{\frac{10}{12}}=\frac{5}{6}$$ *the lowest terms required.*

Explanation.

Here I readily discover by inspection that 6 will divide both terms of the given fraction, and the quotient is $\frac{120}{144}$; this again will divide by 4, and the quotient is $\frac{30}{36}$; this I find will divide by 3, and the quotient is $\frac{10}{12}$; I divide this by 2, and the quotient $\frac{5}{6}$ not being divisible (viz. both terms) by any number greater than 1, is the answer.

If the numbers you divided by, viz. 6, 4, 3, and 2, be multiplied together, the product 144 is the greatest common measure of the given fraction.

33. Reduce $\frac{240}{720}$ to the least terms possible.

$$\overset{10)}{\frac{240}{720}}=\overset{3)}{\frac{24}{72}}=\overset{4)}{\frac{8}{24}}=\overset{2)}{\frac{2}{6}}=\frac{1}{3}$$ *the Answer.*

34. Reduce $\frac{296400}{1284000}$ to its least terms,

Here dividing successively by 100, and 12, the Ans. is $\frac{247}{1070}$.

---

between them, and it is required to divide them by any number, *each of the numbers must be divided.* But if they have the sign $\times$ of multiplication between them, then *only one of the numbers must be divided.* Thus, $\frac{3+6+9-12}{3}=1+2+3-4=2$, where *each* number is divided; and $\frac{3\times6\times9\times12}{3}=1\times6\times9\times12$, or $3\times2\times9\times12$, or $3\times6\times3\times12$, or $3\times6\times9\times4\ (=648)$, where *one factor only* in each case is divided.

p This rule is nothing more than reducing an integral quantity into such parts as the denominator expresses: thus in ex. 37 we are required to reduce $4\frac{5}{6}$; now here we bring the 4 into *sixths*, viz. 24 sixths, to this we add the 5, which is likewise sixths, making in the whole 29 sixths, or $\frac{29}{6}$; and the like in all cases: wherefore the rule is plain.

35. Reduce $\frac{192}{576}$ to its lowest terms. *Ans.* $\frac{1}{3}$.

36. Reduce $\frac{2240}{3360}$ to its lowest terms. *Ans.* $\frac{2}{3}$.

172. *To reduce a mixed number to its equivalent improper fraction.*

RULE I. Multiply the whole number by the denominator of the fraction, and to the product add the numerator.

II. Place this number over the denominator of the fraction, and it will be the answer required[q].

37. Reduce $4\frac{5}{6}$ to its equivalent improper fraction.

OPERATION.

$4 \times 6 + 5 = 29$ *numerator.*

Then $\frac{29}{6}$ *the answer.*

Explanation.

I multiply the whole number 4 by the denominator 6, and to the product 24 I add the numerator 5, which makes 29; this I place over the denominator 6 for the answer.

38. Reduce $5\frac{6}{7}$ and $8\frac{9}{10}$ to improper fractions.

$5\frac{6}{7} = \frac{5 \times 7 + 6}{7} = \frac{41}{7}$ *Ans.* $\quad 8\frac{9}{10} = \frac{8 \times 10 + 9}{10} = \frac{89}{10}$ *Ans.*

39. Reduce $12\frac{3}{5}$ to its equivalent improper fraction. *Ans.* $\frac{63}{5}$.

40. Reduce $1234\frac{5}{6}$ to an improper fraction. *Ans.* $\frac{7409}{6}$.

41. Reduce $312\frac{114}{115}$ to an improper fraction. *Ans.* $\frac{35994}{115}$.

173. *To reduce an improper fraction to its equivalent whole or mixed number.*

RULE I. Divide the numerator by the denominator, and the quotient will be the whole number.

II. Place the remainder (if any) over the denominator, and it will be the fraction, which must be subjoined to the whole number for the answer[q].

---

[q] This mode of operating is the converse of the former. Here we have a number of *parts* given, and are required to find how many *wholes* can be made out of them, supposing that one *whole* contains as many parts as are expressed by the denominator; in order to this, we must evidently divide the numerator

42. Reduce $\frac{29}{6}$ to its equivalent mixed number.

OPERATION. *Explanation.*

$\frac{29}{6} = 4\frac{5}{6}$ *Ans.* Here I divide the numerator 29 by the denominator 6, and the quotient is the whole number; also the remainder 5 placed over the denominator 6 gives $\frac{5}{6}$ for the fraction; these connected give $4\frac{5}{6}$ for the answer.

43. Reduce $\frac{41}{7}$ and $\frac{89}{10}$ to mixed numbers.

$\frac{41}{7} = 5\frac{6}{7}$ *Ans.* $\frac{89}{10} = 8\frac{9}{10}$ *Ans.*

44. Reduce $\frac{72}{7}$ and $\frac{100}{9}$ to mixed numbers. *Ans.* $10\frac{2}{7}$ *and* $11\frac{1}{9}$.

45. Reduce $\frac{234}{12}$ and $\frac{700}{13}$ to mixed numbers. *Ans.* $19\frac{6}{12}$ *and* $53\frac{11}{13}$.

46. Reduce $\frac{35994}{115}$ to a mixed number. *Ans.* $312\frac{114}{115}$.

174. *To reduce a whole number to an equivalent fraction, having a given denominator.*

RULE. Multiply the whole number by the given denominator, and under the product place the said denominator, and it will be the fraction required[r].

47. Reduce 3 to an equivalent fraction, having 4 for a denominator.

$$Thus,\ 3 = \frac{3 \times 4}{4} = \frac{12}{4}\ the\ answer.$$

---

by the denominator, the quotient will then express the number of *wholes* contained in the given fraction, and the remainder will be the *parts* over; this is plain from ex. 42. This rule and the preceding prove each other.

[r] Here the given number is reduced into such *parts* as are denoted by the denominator; under these the denominator is placed, to designate those parts. The truth of the rule may be shewn by dividing the numerator of this fraction by its denominator, by which the given number will be produced. Hence any whole number is reduced to the form of a fraction, by placing 1 under it for a denominator.

48. Reduce 8 and 9 to equal fractions, having 10 for the denominator of each.

First, $8 = \frac{8 \times 10}{10} = \frac{80}{10}$ *Ans.* *Secondly*, $9 = \frac{9 \times 10}{10} = \frac{90}{10}$ *Ans.*

49. Reduce 2, 3, and 4 to equal fractions, having 25 for a denominator. *Ans.* $\frac{50}{25}$, $\frac{75}{25}$, *and* $\frac{100}{25}$.

50. Reduce 9, 8, 7, 6 to fractions, which will have 5, 4, 3, and 2 respectively for denominators. *Ans.* $\frac{45}{5}$, $\frac{32}{4}$, $\frac{21}{3}$, *and* $\frac{12}{2}$.

175. *To reduce a compound fraction to its equivalent simple one, or to a simpler form.*

RULE. Multiply all the numerators continually together for a numerator, and the denominators together for a denominator. Reduce this new fraction, if it be a proper fraction and requires it, to its lowest terms, (by Art. 170. 171.); but if the new fraction be an improper one, reduce it to a whole or mixed number, by Art. 173 *.

51. Reduce $\frac{2}{3}$ of $\frac{5}{7}$ to a simple fraction.

Thus $\frac{2}{3}$ *of* $\frac{5}{7} = \frac{13}{20}$ *the answer. Here we multiply* 2 *and* 5 *together for the numerator, and* 3 *and* 7 *together for the denominator.*

---

* Having a part of a *part* given, we are here taught how to reduce it to its proper part of the *whole*. Let $\frac{2}{3}$ of $\frac{3}{4}$ be a compound fraction proposed to be reduced; now $\frac{1}{3}$ of $\frac{3}{4}$ is evidently $\frac{1}{4}$, wherefore $\frac{2}{3}$ of $\frac{3}{4}$ will be twice as great, or $\frac{2}{4}$, or $\frac{1}{2}$, which is the same as the rule; for by it $\frac{2}{3}$ of $\frac{3}{4} = \frac{6}{12} = \frac{1}{2}$, the very same as before. Again, $\frac{3}{5}$ of $\frac{2}{7}$ will evidently be $\frac{6}{35}$; for $\frac{1}{5}$ of $\frac{2}{7} = \frac{2}{7} \div 5 = \frac{2}{35}$; therefore $\frac{3}{5}$ of $\frac{2}{7} = \frac{2}{35} \times 3 = \frac{6}{35}$ as before: whence the rule is manifest.

52. Reduce $\frac{4}{5}$ of $\frac{2}{3}$ of $\frac{6}{7}$ to a simple fraction.

Thus $\frac{4}{5}$ *of* $\frac{2}{3}$ *of* $\frac{6}{7}=\frac{48}{105}=\frac{16}{35}$ *the answer. Here we reduce the fraction* $\frac{48}{105}$ *to its lowest terms by Art.* 170, 171.

53. Reduce $\frac{4}{5}$ of $\frac{3}{2}$ of $\frac{9}{2}$ of $\frac{3}{4}$ to a simpler form.

Thus $\frac{4}{5}$ *of* $\frac{3}{2}$ *of* $\frac{9}{2}$ *of* $\frac{3}{4}=\frac{324}{80}=\frac{81}{20}=4\frac{1}{20}$ *the answer.*

Here we reduce $\frac{324}{80}$ *to its lowest terms* $\frac{81}{20}$ *by Art.* 170, 171, *and then we reduce this improper fraction to a mixed number, Art.* 173.

54. Reduce $\frac{1}{3}$ of $\frac{5}{7}$ of $\frac{4}{9}$ to a simple fraction. *Ans.* $\frac{20}{189}$.

55. Reduce $\frac{4}{7}$ of $\frac{7}{9}$ of $\frac{1}{4}$ of $\frac{3}{5}$ to a simple fraction. *Ans.* $\frac{1}{15}$.

56. Reduce $\frac{3}{7}$ of $\frac{8}{3}$ of $\frac{5}{4}$ of $\frac{7}{8}$ to a simpler form. *Answer* $\frac{5}{4}=1\frac{1}{4}$.

176. *When any of the terms are whole or mixed numbers,* they must first be reduced to improper fractions by Art. 172, and then proceed as before[1].

57. Reduce $\frac{4}{5}$ of $\frac{3}{7}$ of $5\frac{1}{4}$ to a simpler form.

Here $5\frac{1}{4}=\frac{21}{4}$ *by Art.* 172.

Therefore $\frac{4}{5}$ *of* $\frac{3}{7}$ *of* $\frac{21}{4}=\frac{252}{140}=\frac{63}{35}=\frac{9}{5}=1\frac{4}{5}$ *Ans.*

58. Reduce $\frac{1}{7}$ of $\frac{2}{9}$ of $\frac{1}{12}$ of $1\frac{7}{8}$ to a simple fraction.

Here $1\frac{7}{8}=\frac{15}{8}$ *by Art.* 172.

Therefore $\frac{1}{7}$ *of* $\frac{2}{9}$ *of* $\frac{1}{12}$ *of* $\frac{15}{8}=\frac{30}{6048}=\frac{5}{1008}$ *Ans.*

[1] The reason of this rule is sufficiently plain from the foregoing notes.

59. Reduce $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{7}{9}$ of $2\frac{3}{4}$ to a simpler form. *Ans.* $\frac{77}{60}$, *or* $1\frac{17}{60}$.

60. Reduce $\frac{2}{7}$ of $\frac{3}{6}$ of $\frac{4}{9}$ of $12\frac{1}{4}$ to a simpler form. *Ans.* $\frac{20}{21}$.

177. *To reduce a complex fraction to its equivalent simple one. First, when one term only of the given fraction is a mixed number.*

RULE I. Reduce the mixed term to an improper fraction, and make the numerator of this improper fraction that term of a new fraction which corresponds with the mixed term of the given one.

II. Multiply the unmixed term of the given fraction by the denominator of the mixed one, and make the product the other term of the said new fraction, which will be the fraction required [v].

61. Reduce $\frac{2\frac{3}{4}}{5}$ to its equivalent simple fraction.

OPERATION.

$$\frac{2\frac{3}{4}}{5} = \frac{2 \times 4 + 3}{5 \times 4} = \frac{11}{20} \textit{ Ans.}$$

Explanation.

Here I reduce the numerator $2\frac{3}{4}$ to an improper fraction, the numerator of which 11 I make the *numerator* of a new fraction; and the denominator 4 I multiply into the unmixed part 5, making 20 for the denominator of the new fraction.

---

[v] The numerator and denominator are here equally multiplied; wherefore the resulting fraction will, it is plain, be equal to the given one: thus, ex. 61. $2\frac{3}{4}$ and 5 are both multiplied by 4; for taking away the denominator from $\frac{3}{4}$ multiplies it by 4; wherefore multiplying the 2 by 4 produces 8, and multiplying the 5 by 4 produces 20, whence the fraction becomes $\frac{8+3}{20}$ or $\frac{11}{20}$, as in the example.

62. Reduce $\frac{3}{4\frac{5}{6}}$ to its equivalent simple fraction.

OPERATION.

$$\frac{3}{4\frac{5}{6}} = \frac{3 \times 6}{4 \times 6 + 5} = \frac{18}{29} \textit{ Ans.}$$

Explanation.

Here the denominator being the mixed part, I reduce it to an improper fraction, and make its numerator 29 the *denominator* of a new fraction; I multiply its denominator 6 into the unmixed term 3 for the numerator of the new fraction $\frac{18}{29}$, which is the answer.

63. Reduce $\frac{5\frac{1}{2}}{6}$ to a simple fraction.

Here $\frac{5\frac{1}{2}}{6} = \frac{5 \times 2 + 1}{6 \times 2} = \frac{11}{12}$ *the answer.*

64. Reduce $\frac{7}{8\frac{9}{10}}$ to a simple fraction.

Here $\frac{7}{8\frac{9}{10}} = \frac{7 \times 10}{8 \times 10 + 9} = \frac{70}{89}$ *the answer.*

65. Reduce $\frac{4\frac{6}{7}}{5}$ to a simple fraction. *Ans.* $\frac{34}{35}$.

66. Reduce $\frac{1}{2\frac{1}{2}}$ to a simple fraction. *Ans.* $\frac{2}{5}$.

67. Reduce $\frac{12\frac{4}{5}}{20}$ to a simple fraction. *Ans.* $\frac{16}{25}$.

178. *Secondly, when both terms are mixed numbers.*

RULE I. Reduce each of the terms to an improper fraction, and place the numerator of each opposite that term of the given fraction from whence it is derived, and a new fraction will be formed.

II. Multiply the numerator of this new fraction by the denominator of the *lower* term of the given complex fraction for a numerator, and the denominator of the new fraction by the denominator of the *upper* term for a denominator, and it will be the fraction required[*].

---

[*] The numerator and denominator are equally here multiplied, and therefore the resulting fraction is equal to the given one; thus in ex. 68. both terms are multiplied by 5 × 8; whence the rule is evident.

68. Reduce $\dfrac{3\frac{4}{5}}{6\frac{7}{8}}$ to a simple fraction.

OPERATION.

First, $\dfrac{3 \times 5 + 4}{6 \times 8 + 7} = \dfrac{19}{55}$ *new fraction.*

Then, $\dfrac{19 \times 8}{55 \times 5} = \dfrac{152}{275}$ *the answer.*

Or thus more conveniently.

$\dfrac{3\frac{4}{5}}{6\frac{7}{8}} = \dfrac{(3 \times 5 + 4) \times 8}{(6 \times 8 + 7) \times 5} = \dfrac{152}{275}$ *the ans.*

Explanation.

I first reduce $3\frac{4}{5}$ to an improper fraction, the numerator of which is 19; next I reduce $6\frac{7}{8}$ to an improper fraction, the numerator of which is 55; these form the new fraction $\frac{19}{55}$: I then multiply the numerator 19 by the denominator 8, and the denominator 55 by the denominator 5, for the answer. The second method differs only in form from the first.

69. Reduce $\dfrac{12\frac{6}{7}}{13\frac{3}{4}}$ to a simple fraction.

Thus, $\dfrac{12 \times 7 + 6}{13 \times 4 + 3} = \dfrac{90}{55} = \dfrac{18}{11}$ *new fraction; then* $\dfrac{18 \times 4}{11 \times 7} = \dfrac{72}{77}$ *Ans.*

Or thus, $\dfrac{12\frac{6}{7}}{13\frac{3}{4}} = \dfrac{(12 \times 7 + 6) \times 4}{(13 \times 4 + 3) \times 7} = \dfrac{360}{385} = \dfrac{72}{77}$ *Ans. as before.*

70. Reduce $\dfrac{123\frac{12}{13}}{234\frac{15}{16}}$ to a simple fraction.

Thus, $\dfrac{123\frac{12}{13}}{234\frac{15}{16}} = \dfrac{(123 \times 13 + 12) \times 16}{(234 \times 16 + 15) \times 13} = \dfrac{25776}{48867} = \dfrac{8592}{16289}$ *Ans.*

71. Reduce $\dfrac{2\frac{3}{4}}{5\frac{6}{7}}$ and $\dfrac{1\frac{2}{3}}{2\frac{3}{5}}$ to simple fractions. *Ans.* $\dfrac{77}{164}$ *and* $\dfrac{25}{39}$.

179. When either of the terms of a complex fraction is a simple fraction, or when both terms are simple, the operation will be somewhat less difficult[y]; thus,

[y] Here both terms of the fraction are equally multiplied: thus in ex. 72. both terms are multiplied by 7×3, wherefore the resulting fraction $\frac{14}{15}$ equals the given one, which was to be shewn; and the same is true in every instance of this kind.

72. Reduce $\frac{\frac{2}{3}}{\frac{5}{7}}$ to a simple fraction.

Multiply 2 *by* 7, *and* 5 *by* 3, *thus,* $\frac{2 \times 7}{5 \times 3} = \frac{14}{15}$ *Ans.*

73. Reduce $\frac{\frac{7}{9}}{2\frac{1}{4}}$ to a simple fraction.

Here $\frac{\frac{7}{9}}{2\frac{1}{4}} = \frac{7 \times 4}{(2 \times 4 + 1) \times 9} = \frac{28}{81}$ *Ans.*

74. Reduce $\frac{\frac{3}{5}}{\frac{4}{7}}$, $\frac{\frac{2}{3}}{1\frac{1}{5}}$, and $\frac{3\frac{1}{7}}{\frac{2}{3}}$, to simple fractions. *Ans.* $\frac{21}{20} = 1\frac{1}{20}$, $\frac{10}{18} = \frac{5}{9}$, *and* $\frac{66}{14} = \frac{33}{7} = 4\frac{5}{7}$.

180. *To reduce fractions of different denominators to equivalent fractions, having a common denominator.*

RULE I. Multiply each numerator into all the denominators except its own for a new numerator.

II. Multiply all the denominators continually together for a common denominator.

III. Place each new numerator over the common denominator, and there will be as many new fractions formed as there are fractions given in the question; also each of the new fractions will be equal to its respective one in the question, viz. the first equal to the first, the second to the second, &c.[a]

---

[a] It will appear (by placing the numbers in a convenient manner) that both terms of each fraction are equally multiplied; thus in ex. 76. each of the terms of $\frac{2}{7}$ is multiplied by 5×6, each of the terms of $\frac{4}{5}$ by 7×6, and each of the terms of $\frac{1}{6}$ by 7×5, as follows:

$$\left.\begin{array}{l} \frac{2}{7} \left| \frac{\times 5 \times 6}{\times 5 \times 6} = \frac{60}{210} \right. \\ \frac{4}{5} \left| \frac{\times 7 \times 6}{\times 7 \times 6} = \frac{168}{210} \right. \\ \frac{1}{6} \left| \frac{\times 7 \times 5}{\times 7 \times 5} = \frac{35}{210} \right. \end{array}\right\} \text{the answers as in the example; wherefore the rule is manifest.}$$

75. Reduce $\frac{4}{5}$ and $\frac{6}{7}$ to equal fractions, having a common denominator.

OPERATION.

$$\left.\begin{array}{l}4 \times 7 = 28 \\ 6 \times 5 = 30\end{array}\right\} \textit{new numerators.}$$

$5 \times 7 = 35$ *com. denominator.*

Wherefore $\frac{28}{35}$ *and* $\frac{30}{35}$ *the Ans.*

Explanation.

Here I multiply the first numerator 4 into the denominator 7, which gives 28 for the first new numerator. I next multiply the second numerator 6 into the denominator 5, and the product 30 is the second new numerator. Next I multiply the denominators 5 and 7 together, and the product 35 is the common denominator. Lastly, I place both new numerators over the common denominator for the answer.

76. Reduce $\frac{2}{7}$, $\frac{4}{5}$, and $\frac{1}{6}$, to a common denominator.

OPERATION.

$$\left.\begin{array}{l}2 \times 5 \times 6 = 60 \\ 4 \times 7 \times 6 = 168 \\ 1 \times 7 \times 5 = 35\end{array}\right\} \textit{new num.}$$

$7 \times 5 \times 6 = 210$ *com. denom.*

Ans. $\frac{60}{210}$, $\frac{168}{210}$, *and* $\frac{35}{210}$.

Explanation.

Here I multiply the numerator 2 into the denominators 5 and 6; the numerator 4 into the denominators 7 and 6; and the numerator 1 into the denominators 7 and 5; the products are the new numerators; I then multiply the denominators 7, 5, and 6 together for the common denominator.

77. Reduce $\frac{2}{3}$, $\frac{4}{7}$, $\frac{3}{5}$, and $\frac{1}{2}$, to a common denominator.

$$\textit{Thus, } \left.\begin{array}{l}2 \times 7 \times 5 \times 2 = 140 \\ 4 \times 3 \times 5 \times 2 = 120 \\ 3 \times 3 \times 7 \times 2 = 126 \\ 1 \times 3 \times 7 \times 5 = 105\end{array}\right\} \textit{new numerators.}$$

$3 \times 7 \times 5 \times 2 = 210$ *common denominator.*

Ans. $\frac{140}{210}$, $\frac{120}{210}$, $\frac{126}{210}$, *and* $\frac{105}{210}$.

Whole and mixed numbers must be reduced to improper fractions, compound and complex fractions to simple ones, when any of these occur in the question.

78. Reduce $2\frac{3}{4}$, $\frac{2}{3}$ of $\frac{4}{5}$, and $\frac{1\frac{1}{2}}{2\frac{2}{3}}$, to a common denominator.

Here $2\frac{3}{4} = \frac{2 \times 4 + 3}{4} = \frac{11}{4}$ *by Art.* 172. $\frac{2}{3}$ *of* $\frac{4}{5} = \frac{8}{15}$ *by Art.* 175. *and* $\frac{1\frac{1}{2}}{2\frac{2}{3}} = \frac{(1 \times 2 + 1) \times 3}{(2 \times 3 + 2) \times 2} = \frac{9}{16}$ *by Art.* 178.

Whence the fractions to be reduced are $\frac{11}{4}$, $\frac{8}{15}$, *and* $\frac{9}{16}$.

Wherefore $\left.\begin{array}{l} 11 \times 15 \times 16 = 2640 \\ 8 \times 4 \times 16 = 512 \\ 9 \times 4 \times 15 = 540 \end{array}\right\}$ *new numerators.*

$4 \times 15 \times 16 = 960$ *common denominator.*

Ans. $\frac{2640}{960}$, $\frac{512}{960}$, *and* $\frac{540}{960}$.

79. Reduce $\frac{2}{3}$ and $\frac{4}{5}$ to equal fractions, having a common denominator. *Ans.* $\frac{10}{15}$ *and* $\frac{12}{15}$.

80. Reduce $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{8}{9}$, to a common denominator. *Ans.* $\frac{189}{315}$, $\frac{180}{315}$, *and* $\frac{280}{315}$.

81. Reduce $\frac{2}{7}$, $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{4}{9}$, to a common denominator. *Ans.* $\frac{270}{945}$, $\frac{315}{945}$, $\frac{378}{945}$, *and* $\frac{420}{945}$.

82. Reduce $\frac{7}{8}$, $\frac{3}{4}$, $\frac{2}{7}$, and $\frac{1}{4}$, to a common denominator. *Ans.* $\frac{784}{896}$, $\frac{672}{896}$, $\frac{256}{896}$, *and* $\frac{224}{896}$.

83. Reduce $\frac{1}{2}$ of $\frac{5}{7}$, $1\frac{1}{4}$, and $\frac{2\frac{1}{4}}{3\frac{1}{2}}$, to a common denominator. *Ans.* $\frac{1120}{3136}$, $\frac{2240}{3136}$, $\frac{5880}{3136}$, *and* $\frac{2016}{3136}$.

181. *To reduce fractions to other equivalent ones, having the least common denominator.*

RULE I. Find the least common multiple of all the denominators by Art. 169, and it will be the common denominator required.

II. Divide the common denominator by the denominator of each fraction, and multiply the quotient by the numerator, the several products will be the numerators, which place over the common denominator for the answer *.

84. Reduce $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{11}{12}$, to equivalent fractions, having the least common denominator.

OPERATION.

The least common multiple of 4 *and* 8 *by Art.* 168 *is* $\frac{4 \times 8}{4} =$ 8, *and of* 8 *and* 12 *is* $\frac{8 \times 12}{4}$ = 24 *the common denominator.*

$$\left.\begin{array}{l} \textit{Therefore } \frac{24}{4} \times 3 = 18 \\ \frac{24}{8} \times 5 = 15 \\ \frac{24}{12} \times 11 = 22 \end{array}\right\} \textit{new num.}$$

Whence $\frac{18}{24}$, $\frac{15}{24}$, *and* $\frac{22}{24}$, *the answer.*

Explanation.

I first find the least common multiple of the denominators 4 and 8, which is 8; then of 8 and the remaining denominator 12, which is 24; this being the least common multiple of 4, 8, and 12, viz. of all the denominators, is the least common denominator. I next divide 24 by 4, 8, and 12, the three denominators, and multiply the quotients by 3, 5, and 11, the three numerators, and the products 18, 15, and 22, placed over the common denominator 24, give the answer.

---

* The common denominator is a common multiple of all the given denominators, as is plain; and the process, as directed in the second part of the rule, is merely the taking such parts of it for new numerators as have the same ratio respectively to the whole as the numerator of each given fraction has to its denominator; thus, ex. 84. multiplying 24 (the common denominator) by 3, and dividing by 4, is the same as taking $\frac{3}{4}$ of 24, the result of which, viz. 18, has the same ratio to 24 that 3 has to 4; whence $\frac{18}{24}$ = the given fraction $\frac{3}{4}$. In like manner 15 : 24 :: 5 : 8, or $\frac{15}{24} = \frac{5}{8}$. Likewise 22 : 24 :: 11 : 12; wherefore $\frac{22}{24} = \frac{11}{12}$; from whence the reason of the rule will be understood.

85. Reduce $\frac{4}{9}$, $\frac{5}{8}$, $\frac{5}{6}$, and $\frac{3}{4}$, to the least common denominator.

First, $\frac{9 \times 8}{1} = 72$, *then* $\frac{72 \times 6}{6} = 72$, *and* $\frac{72 \times 4}{4} = 72$, *the least common multiple of* 9, 8, 6, *and* 4, *by Art.* 169.

PROSS 236
72÷9=
8×4=32

$$\left.\begin{array}{l} \textit{Then } \frac{72}{9} \times 4 = 32 \\ \frac{72}{8} \times 5 = 45 \\ \frac{72}{6} \times 5 = 60 \\ \frac{72}{4} \times 3 = 54 \end{array}\right\} \textit{new numerators.}$$

Wherefore $\frac{32}{72}$, $\frac{45}{72}$, $\frac{60}{72}$, *and* $\frac{54}{72}$, *the answer.*

86. Reduce $\frac{6}{7}$, $\frac{2}{3}$, $\frac{5}{9}$, and $\frac{3}{14}$, to the least common denominator.

First, $\frac{7 \times 3}{1} = 21$, $\frac{21 \times 9}{3} = 63$, $\frac{63 \times 14}{7} = 126$ *common denominator.*

Then $\frac{126}{7} \times 6 = 108$, $\frac{126}{3} \times 2 = 84$, $\frac{126}{9} \times 5 = 70$,

and $\frac{126}{14} \times 3 = 27$, *the new numerators.*

Wherefore $\frac{108}{126}$, $\frac{84}{126}$, $\frac{70}{126}$, *and* $\frac{27}{126}$, *the answer.*

87. Reduce $\frac{5}{6}$ and $\frac{8}{9}$ to the least common denominator.

Ans. $\frac{15}{18}$ *and* $\frac{16}{18}$.

88. Reduce $\frac{3}{4}$, $\frac{7}{10}$, and $\frac{11}{12}$, to the least common denominator. *Ans.* $\frac{45}{60}$, $\frac{42}{60}$, *and* $\frac{55}{60}$.

Error

89. Reduce $\frac{11}{12}$, $\frac{3}{14}$ and $\frac{5}{16}$, to the least common denominator. *Ans.* $\frac{308}{336}$, $\frac{72}{336}$, *and* $\frac{105}{336}$.

90. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{8}{9}$, to the least common denominator. *Ans.* $\frac{63}{126}$, $\frac{84}{126}$, $\frac{90}{126}$, *and* $\frac{112}{126}$.

182. *To reduce any fraction to another of equal value, having a given denominator.* ok

RULE I. Find the numerator by a Rule of Three stating; thus, say, as the denominator of the given fraction : is to its numerator :: so is the given denominator : to a fourth number, which will be the required numerator.

II. Place this numerator over the given denominator, and it will give the fraction required [b].

91. Reduce $\frac{4}{5}$ to an equal fraction, having 100 for its denominator.

Thus, 5 *den.* : 4 *num.* :: 100 *den.* : $\frac{4 \times 100}{5} = \frac{400}{5} =$ 80 *the numerator.*

Therefore $\frac{80}{100}$ *is the fraction required.*

92. Reduce $\frac{11}{12}$ to a fraction of equal value, having 20 for its denominator.

Thus 12 *den.* : 11 *num.* :: 20 *den.* : $\frac{11 \times 20}{12} = \frac{220}{12} =$ $18\frac{4}{12} = 18\frac{1}{3}$ *numerator.*

Therefore $\frac{18\frac{1}{3}}{20}$ *the fraction required.*

93. Reduce $\frac{2}{3}$ to a fraction of equal value, whose denominator is 27. *Ans.* $\frac{18}{27}$.

94. Reduce $\frac{19}{20}$ to an equal fraction, having 5 for its denominator. *Ans.* $\frac{4\frac{3}{4}}{5}$.

---

[b] Here the numerator of the given fraction is to its denominator as the numerator of the resulting fraction is to its denominator; therefore the latter fraction is equal to the former; which shews the rule to be right.

183. *To reduce the fraction of a less denomination to that of a greater, retaining the same value.*

RULE. Multiply the denominator by all the denominations between that which is given and that which is required; over the product place the given numerator, and you will have the fraction required[c].

95. Reduce $\frac{8}{9}$ of a penny to the fraction of a pound.

OPERATION.

$$Thus,\ \frac{8}{9}\ d. = \frac{8}{9 \times 12 \times 20} = \frac{8}{2160} = \frac{1}{270}\ Ans.$$

Explanation.

I multiply the denominator 9 by 12 and 20, (because 12 pence make a shilling, and 20 shillings a pound,) and place the numerator 8 over, which gives $\frac{8}{2160}$, this reduced to its lowest terms is $\frac{1}{270}$ the answer.

96. Reduce $\frac{4}{5}$ of a grain to the fraction of a pound troy.

$$Thus,\ \frac{4}{5}\ gr. = \frac{4}{5 \times 24 \times 20 \times 12} = \frac{4}{28800} = \frac{1}{7200}\ lb.\ Ans.$$

97. Reduce $\frac{2}{3}$ of a pound to the fraction of a cwt.

$$Thus,\ \frac{2}{3}\ lb. = \frac{2}{3 \times 28 \times 4} = \frac{2}{336} = \frac{1}{168}\ cwt.\ Ans.$$

98. Reduce $\frac{6}{7}$ of a pole to the fraction of a mile.

$$Thus,\ \frac{6}{7}\ p. = \frac{6}{7 \times 40 \times 8} = \frac{6}{2240} = \frac{3}{1120}\ mile,\ the\ Ans.$$

99. Reduce $\frac{1}{2}$ of a penny to the fraction of a pound. *Ans.* $\frac{1}{480}$ *L.*

---

[c] This rule is equivalent to that for reducing a compound fraction to a simple one (Art. 175.); thus, ex. 95, $\frac{8}{9}$ of a penny is $\frac{8}{9}$ of $\frac{1}{12}$ of $\frac{1}{20}$ of a pound, that is $\frac{8}{9 \times 12 \times 20}$; and the same in general: wherefore the reason of the process is plain.

100. Reduce $\frac{4}{5}$ of a penny to the fraction of a guinea. *Ans.* $\frac{1}{315}$ *guinea.*

101. Reduce $\frac{9}{10}$ of a nail to the fraction of a yard. *Answer* $\frac{9}{160}$ *yard.*

102. Reduce $\frac{7}{8}$ of a square pole to the fraction of an acre. *Ans.* $\frac{7}{1280}$ *acre.*

103. Reduce $\frac{2}{3}$ of a bushel of coals to the fraction of a chaldron. *Ans.* $\frac{1}{54}$ *chaldron.*

104. Reduce $\frac{9}{10}$ of a day to the fraction of a year. *Answer* $\frac{9}{3650}$ *year.*

184. *To reduce the fraction of a greater denomination to that of a less, retaining the same value.*

RULE. Multiply the numerator by all the denominations between the given one and that which is required; under the product place the given denominator, and it will give the fraction required[d].

105. Reduce $\frac{1}{270}$ of a pound to the fraction of a penny.

OPERATION.

$$Thus,\ \frac{1 \times 20 \times 12}{270} = \frac{240}{270} = \frac{8}{9}\ d.\ Ans.$$

Explanation.

I multiply the numerator 1 by 20 and 12, (the denominations between pounds and pence,) and place the denominator 270 under the product, making the fraction $\frac{240}{270}$, which reduced to its lowest terms gives $\frac{8}{9}$.

---

[d] This rule is likewise of the same nature with Art. 175, for its operation is simply the reducing a compound fraction to a simple one; thus (ex. 105.) $\frac{1}{270}$ of a pound is $\frac{1}{270}$ of $\frac{20}{1}$ of $\frac{12}{1}$ of a penny, or $\frac{1 \times 20 \times 12}{270}$; wherefore the rule is evident.

106. Reduce $\frac{1}{7200}$ of a pound troy to the fraction of a grain.

$$Thus, \frac{1 \times 12 \times 20 \times 24}{7200} = \frac{5760}{7200} = \frac{4}{5} \text{ gr. Ans.}$$

107. Reduce $\frac{9}{160}$ of a yard to the fraction of a nail.

$$Thus, \frac{9 \times 4 \times 4}{160} = \frac{144}{160} = \frac{9}{10} \text{ n. Ans.}$$

108. Reduce $\frac{1}{168}$ of a cwt. to the fraction of a lb. *Ans.* $\frac{2}{3}$ *lb.*

109. Reduce $\frac{1}{960}$ of a pound to the fraction of a penny. *Ans.* $\frac{1}{4}$ *d.*

110. Reduce $\frac{1}{504}$ of a guinea to the fraction of a penny. *Ans.* $\frac{1}{2}$ *d.*

111. Reduce $\frac{40}{147}$ of a guinea to the fraction of a pound. *Ans.* $\frac{2}{7}$ *L.*

185. *To reduce compound numbers, &c. in money, weights, and measures, to fractions of some higher denomination.*

RULE I. Reduce the given number to the lowest denomination mentioned for a numerator.

II. Reduce the integer of which the above is to be made a fraction into the same denomination for a denominator.

III. Place the numerator over the denominator, and reduce the fraction to its lowest terms[e]. Art. 170.

---

[e] In ex. 112, 3*s.* 6*d.* = 42 pence, and 1*l.* = 240 pence; therefore 3*s.* 6*d.* is 42 parts out of 240 of a pound; whence the reason of this process is plain.

It is best to reduce the given number, and the integer of which it is to be made a fraction, to the *greatest* denomination common to both; for then the fraction will be in its lowest terms. Thus (ex. 112.) 3*s.* 6*d.* = 7 sixpences, and 1*l.* = 40 sixpences; therefore 3*s.* 6*d.* = $\frac{7}{40}$*l.* as in the example.

112. **Reduce 3s. 6d. to the fraction of a pound.**

Thus, 3s. 6d. = 42 *pence, the numerator.*

And 1 *pound* = 240 *pence, the denominator.*

$$\text{Therefore } \frac{42}{240} = \frac{7}{40} \text{ l. the answer.}$$

113. **Reduce 14s. 6d.½ to the fraction of a guinea.**

Thus, 14*s.* 6*d.*½ = 349 *halfpence, the numerator.*

And 1 *guinea* = 504 *halfpence, the denominator.*

$$\text{Therefore } 14s.\ 6d.\tfrac{1}{2} = \frac{349}{504} \text{ of a guinea, Ans.}$$

114. Reduce 1*oz.* 2*dwts.* to the fraction of a lb. troy.

Thus, 1*oz.* 2*dwts.* = 22*dwts. the numerator.*

And 1*lb.* = 240*dwts. the denominator.*

$$\text{Therefore } 1oz.\ 2dwts. = \frac{22}{240} = \frac{11}{120} lb. \text{ troy, Ans.}$$

115. Reduce 1*lb.* 2*oz.* 3½*dr.* to the fraction of a lb.

Thus, 1*lb.* 2*oz.* 3½*dr.* = 583 *half drams, numerator.*

And 1*lb.* = 512 *half drams, denominator.*

$$\text{Therefore } 1lb.\ 2oz.\ 3\tfrac{1}{2}dr. = \frac{583}{512} lb. \text{ Ans.}$$

116. Reduce 12*s.* to the fraction of a pound. *Ans.* $\frac{3}{5}$ *L.*

117. Reduce 19*s.* 6*d.* to the fraction of a pound. *Ans.* $\frac{39}{40}$ *L.*

118. Reduce 3*qrs.* 14*lb.* to the fraction of a cwt. *Ans.* $\frac{7}{8}$ *cwt.*

119. Reduce 3*qrs.* 3*n.* to the fraction of an English ell. *Ans.* $\frac{3}{4}$ *E. E.*

120. Reduce 1*hhd.* 2*gal.* 3*qts.* to the fraction of a tun. *Ans.* $\frac{263}{1008}$ *tun.*

121. Reduce 3*bu.* 2½*pks.* to the fraction of a quarter. *Ans.* $\frac{25}{58}$ *quarter.*

122. Reduce 10*bu.* 3*pks.* to the fraction of a chaldron. *Ans.* $\frac{43}{144}$ *chaldron.*

123. Reduce 12*w.* 3*d.* 4*h.* 5*m.* 6″ to the fraction of a year of 365 days 6 hours. *Ans.* $\frac{418417}{1753200}$ *year.*

186. *To find the value of a fraction in the known parts of the integer.*

RULE I. Reduce the numerator to the next lower denomination, divide the result by the denominator, and the quotient will be of the said lower denomination.

II. Reduce the remainder to the next denomination lower than the last, divide the result by the denominator, and the quotient will be of this last denomination.

III. Proceed in this manner until you arrive at the lowest denomination, then collect all the quotients together for the answer [f].

124. Required the value of $\frac{7}{8}$ of a pound sterling.

OPERATION.

$$\frac{7}{8}=\frac{7\times 20}{8}=\frac{140}{8}=17s.\ 6d.\ \textit{the answer.}$$

Explanation.

I reduce the numerator 7 to shillings, viz. 140; this I divide by the denominator 8, and the quotient is 17, and 4 over; 4 shillings are 48 pence, eights in 48 will go 6 times.

125. Reduce $\frac{11}{12}$ *L.* to its proper quantity.

Thus, $\frac{11}{12}=\frac{11\times 20}{12}=\frac{220}{12}=$ 18*s.* 4*d. the answer.*

126. Reduce $\frac{4}{5}$ of a lb. troy to its proper quantity.

Thus, $\frac{4}{5}=\frac{4\times 12}{5}=\frac{48}{5}=$ 9*oz.* 12*dwts. the answer.*

127. Reduce $\frac{7}{9}$ of a ton to its proper quantity.

Thus, $\frac{7}{9}=\frac{7\times 20}{9}=\frac{140}{9}=$ 15*cwt.* 2*qr.* 6*lb.* 3*oz.* $8\frac{4}{9}$*dr.*

---

[f] Since the numerator is less than the denominator, the former may be considered as a *remainder*, and the latter as a *divisor*; whence the rule has its reason in the nature of compound division, where every remainder is reduced to the next lower denomination, and being then divided, produces integers of the said denomination. See *Art.* 108 *to* 124.

128. Reduce $\frac{3}{5}$ of a pound sterling to its proper quantity. *Ans.* 12 *shillings.*

129. Reduce $\frac{2}{7}$ of a pound to its proper sum. *Answer* 5*s.* 8*d.*$\frac{1}{7}$ $\frac{2}{7}$.

130. Reduce $\frac{7}{12}$ of a crown to its proper sum. *Ans.* 2*s.* 11*d.*

131. Reduce $\frac{7}{8}$ of a cwt. to its proper quantity. *Answer* 3*qr.* 14*lb.*

132. Reduce $\frac{25}{56}$ of a quarter of corn to its proper quantity. *Ans.* 3*bu.* 2$\frac{2}{7}$*pks.*

133. Reduce $\frac{14}{15}$ of an acre to its proper quantity. *Answer* 3*r.* 29$\frac{1}{3}$*p.*

134. Reduce $\frac{19}{20}$ of a week to its proper time. *Answer* 6*d.* 15*h.* 36*m.*

ADDITION OF VULGAR FRACTIONS.

187. *When simple fractions are to be added together.*

RULE I. Reduce the given fractions to a common denominator, by Art. 180.

II. Add the new numerators together, under their sum place the common denominator, and reduce the fraction to its lowest terms for the answer, by Art. 170.

III. If the resulting fraction be an improper one, reduce it to its equivalent whole or mixed number, by Art. 173[r].

---

[r] As whole numbers of different denominations cannot be added or subtracted, so dissimilar fractions (namely, such as are not like parts of the same whole) cannot; so, in Example 1, $\frac{2}{7}$ cannot be added to $\frac{3}{8}$ until they are reduced to the same denomination: this reduction being performed, $\frac{2}{7}$ becomes $\frac{16}{56}$, and $\frac{3}{8}$ becomes $\frac{21}{56}$; now these two fractions, namely $\frac{16}{56}$ and $\frac{21}{56}$, being

EXAMPLES.

1. Add $\frac{2}{7}$ and $\frac{3}{8}$ together.

OPERATION.

$$\left.\begin{array}{l}2 \times 8 = 16 \\ 3 \times 7 = 21\end{array}\right\} \textit{new numerators.}$$

37 *their sum.*

$7 \times 8 = 56$ *common denom.*

Answer $\frac{37}{56}$.

Explanation.

I first reduce the given fractions to a common denominator; I then add the new numerators 16 and 21 together, and under their sum 37 place 56, the common denominator, which gives the answer.

2. Add $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{3}{8}$ together.

OPERATION.

$$\left.\begin{array}{l}2 \times 4 \times 8 = 64 \\ 3 \times 3 \times 8 = 72 \\ 3 \times 3 \times 4 = 36\end{array}\right\} \textit{new numerators.}$$

172 *sum.*

$3 \times 4 \times 8 = 96$ *common denom.*

Ans. $\frac{172}{96} = 1\frac{76}{96}$ *sum required.*

Explanation.

Having reduced the fractions to a common denominator, and added the new numerators together as before, I place the sum 172 over the common denominator 96, which being an improper fraction, I reduce it to a mixed number for the answer.

3. Add $\frac{2}{3}$ and $\frac{1}{4}$ together. *Sum* $\frac{11}{12}$.

4. Add $\frac{1}{5}$, $\frac{2}{9}$, and $\frac{3}{7}$ together. *Sum* $\frac{263}{315}$.

---

added together, the sum is evidently $\frac{16+21}{56}$, or $\frac{37}{56}$, as in the example; and the same of others.

Nothing can be plainer than the grounds of this process; for since the denominator only indicates what parts the fraction consists of, therefore, when several fractions having the same denominator are proposed, the comparative value of each will be expressed by its numerator. Thus, let the fractions $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$ be proposed; it is plain that $\frac{2}{7}$ is double of $\frac{1}{7}$, and $\frac{3}{7}$ triple of it, and that the sum of all three can be nothing but *sevenths*, that is, it will consist of as many sevenths as there are units in all the numerators taken together; wherefore $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$ added together will amount to $\frac{6}{7}$, which is the rule.

5. Add $\frac{2}{5}$, $\frac{3}{4}$, $\frac{5}{9}$, and $\frac{7}{8}$ together. *Sum* $3\frac{65}{288}$.

6. Add $\frac{3}{8}$, $\frac{2}{5}$, $\frac{7}{9}$, $\frac{2}{7}$, and $\frac{1}{2}$ together. *Sum* $2\frac{853}{2520}$.

188. *When there are whole or mixed numbers to be added.*

RULE I. Add the fractions together by the preceding rule.

II. Add the whole numbers together, and prefix their sum to the sum of the fractions (found by the preceding rule) for the answer[w].

7. Add $3\frac{1}{7}$, $4\frac{2}{9}$, and $5\frac{1}{6}$ together.

Thus
$$\left.\begin{array}{r} 1 \times 9 \times 6 = 54 \\ 2 \times 7 \times 6 = 84 \\ 1 \times 7 \times 9 = 63 \end{array}\right\} \textit{new numerators.}$$
$$\overline{201} \textit{ their sum.}$$

$7 \times 9 \times 6 = 378$ *common denominator.*

$\frac{201}{378} = \frac{67}{126}$ *sum of the fractions, Art.* 187; *also* $3 + 4 + 5 = 12$ *sum of the whole numbers. Wherefore* $12\frac{67}{126}$ *the sum required.*

8. Add $2\frac{3}{4}$ and $3\frac{4}{5}$ together. *Sum* $6\frac{11}{20}$.

9. Add $1\frac{1}{2}$, $2\frac{2}{3}$, and $3\frac{3}{4}$ together. *Sum* $7\frac{11}{12}$.

10. Add $\frac{5}{7}$, $2\frac{7}{8}$, $5\frac{1}{2}$, and 6 together. *Sum* $15\frac{5}{56}$.

189. *When compound or complex fractions are to be added.*

RULE. Reduce the compound and complex fractions to simple ones, reduce their equivalent simple fractions to a common denominator, and proceed as before[x].

---

[w] This rule is evident; for if the sum of the whole numbers be prefixed to the sum of the fractions, it is plain that the result will be the sum of all the given numbers, both whole and fractional.

[x] Fractions cannot be added together until they are first reduced to *simple*

11. Add $\frac{2}{3}$ of $\frac{5}{7}$, $\frac{2\frac{4}{5}}{7}$, and $\frac{1\frac{2}{3}}{2\frac{1}{2}}$ together.

OPERATION.

First, $\frac{2}{3}$ *of* $\frac{5}{7} = \frac{10}{21}$ *by Art.* 175.

Secondly, $\frac{2\frac{4}{5}}{7} = \frac{2 \times 5 + 4}{7 \times 5} = \frac{14}{35}$ *by Art.* 177.

Thirdly, $\frac{1\frac{2}{3}}{2\frac{1}{2}} = \frac{(1 \times 3 + 2) \times 2}{(2 \times 2 + 1) \times 3} = \frac{10}{15} = \frac{2}{3}$, *by Art.* 178.

Fourthly,
$$\left.\begin{aligned} 10 \times 35 \times 3 &= 1050 \\ 14 \times 21 \times 3 &= 882 \\ 2 \times 21 \times 35 &= 1470 \end{aligned}\right\} \textit{new num.}$$
$$3402 \textit{ their sum.}$$
$$21 \times 35 \times 3 = 2205$$

Lastly, $\frac{3402}{2205} = \frac{486}{315} = \frac{162}{105} = \frac{54}{35} = 1\frac{19}{35}$ *the answer.*

Explanation.

First, I reduce the compound fraction $\frac{2}{3}$ of $\frac{5}{7}$ to $\frac{10}{21}$. Secondly and thirdly, I reduce the two complex fractions to $\frac{14}{35}$ and $\frac{2}{3}$, their respective simple ones. Fourthly, I reduce these three fractions, $\frac{10}{21}$, $\frac{14}{35}$, and $\frac{2}{3}$ to a common denominator. Lastly, I reduce the answer $\frac{3402}{2205}$ to its lowest terms by dividing successively by 7, 3, and 3, and then I reduce the improper fraction $\frac{54}{35}$ to a mixed number.

---

fractions of the same whole, and then to *similar parts* of the same whole, namely, to a common denominator; when all this is effected, the sum of the whole is evidently found by adding all the numerators of the reduced fractions together, and placing under the sum the common denominator, as in Art. 187; wherefore the rule is manifest.

12. Add $\frac{3}{4\frac{2}{5}}$, $\frac{1}{2}$ of $\frac{3}{4}$, and $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{1}{2}$ together.

Thus, $\frac{3}{4\frac{2}{5}} = \frac{3 \times 5}{4 \times 5 + 2} = \frac{15}{22}$; $\frac{1}{2}$ *of* $\frac{3}{4} = \frac{3}{8}$; $\frac{2}{3}$ *of* $\frac{4}{5}$ *of* $\frac{1}{2} = \frac{8}{30} = \frac{4}{15}$.

$$\left.\begin{array}{l} \textit{Then } 15 \times 8 \times 15 = 1800 \\ 3 \times 22 \times 15 = 990 \\ 4 \times 22 \times 8 = 704 \end{array}\right\} \textit{new numerators.}$$

$$\overline{3494} \textit{ their sum.}$$

$$22 \times 8 \times 15 = 2640 \textit{ common denom.}$$

$$\textit{Wherefore } \frac{3494}{2640} = \frac{1747}{1320} = 1\tfrac{427}{1320} \textit{ the answer.}$$

13. Add $\frac{3}{4}$ of $\frac{1}{2}$ and $\frac{2}{5}$ of $\frac{3}{7}$ together. *Sum* $\frac{153}{280}$.

14. Add $\frac{4}{7}$ of $\frac{3}{5}$, $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$, and $\frac{1}{3}$ of $\frac{1}{4}$ together. *Ans.* $\frac{53}{60}$.

15. Add $\frac{2\frac{1}{2}}{3}$ and $\frac{2}{3\frac{3}{5}}$ together. *Sum* $1\frac{7}{18}$.

16. Add $\frac{2}{3}$ of $4\frac{1}{2}$ and $\frac{3\frac{1}{3}}{4\frac{1}{4}}$ together. *Sum* $3\frac{40}{51}$.

190. *When the fractions to be added are of different denominations in money, weights, or measures.*

RULE. Reduce the fractions to their proper quantities, by Art. 186. then add the proper quantities together by the rules of Compound Addition[y].

---

[y] It is plain that the fractions treated of in this rule being dissimilar cannot be added together until they are reduced as the rule directs; when this reduction is performed, the remainder of the operation (depending on the rules of Compound Addition) is sufficiently obvious.

17. Add $\frac{5}{6}$ of a pound, $\frac{7}{12}$ of a crown, and $\frac{8}{9}$ of a shilling together.

OPERATION.

$$\begin{array}{llllr} & & & s. & d. \\ \frac{5}{6} = \frac{5 \times 20}{6} = \frac{100}{6} & = & 16 & & 8 \\ \frac{7}{12} = \frac{7 \times 5}{12} = \frac{35}{12} & = & 2 & & 11 \\ \frac{8}{9} = \frac{8 \times 12}{9} = \frac{96}{9} & = & 0 & & 10\frac{2}{3} \\ \textit{Sum required} & & 1 & 0 & 5\frac{2}{3} \end{array}$$

Explanation.

First, I reduce $\frac{5}{6}$*l.* to its proper quantity, which is 16*s.* 8*d.*; next, $\frac{7}{12}$ *cr.* which is 2*s.* 11*d.*; then $\frac{8}{9}$ *shill.* which is $10\frac{2}{3}$*d.*; these three sums I add together, which gives the answer.

18. Add $\frac{11}{12}$ of a ton, $\frac{5}{8}$ cwt. $\frac{4}{7}$ qr. and $\frac{3}{4}$ lb, together.

$$\begin{array}{llllll} & \textit{cwt.} & \textit{qr.} & \textit{lb.} & \textit{oz.} & \textit{dr.} \\ \textit{Thus, } \frac{11}{12}\,\textit{ton} = \frac{11 \times 20}{12} = \frac{220}{12} = & 18 & 1 & 9 & 5 & 5\frac{1}{3} \\ \frac{5}{8}\,\textit{cwt.} = \frac{5 \times 4}{8} = \frac{20}{8} = & 0 & 2 & 14 & 0 & 0 \\ \frac{4}{7}\,\textit{qr.} = \frac{4 \times 28}{7} = \frac{112}{7} = & 0 & 0 & 16 & 0 & 0 \\ \frac{3}{4}\,\textit{lb.} = \frac{3 \times 16}{4} = \frac{48}{4} = & 0 & 0 & 0 & 12 & 0 \\ \textit{Sum required} & 19 & 0 & 12 & 1 & 5\frac{1}{3} \end{array}$$

19. Add $\frac{4}{5}$ of a pound to $\frac{5}{6}$ of a shilling. *Sum* 16*s.* 10*d.*

20. Add $\frac{5}{12}$ of a guinea, $\frac{1}{6}$ of a pound, and $\frac{3}{8}$ of a shilling together. *Sum* 12*s.* 5*d.*$\frac{1}{2}$.

21. Add $\frac{3}{10}$ of an English ell to $\frac{7}{8}$ of a yard. *Sum* 1*yd.* 1*qr.*

22. Add $\frac{7}{8}$ cwt. and $\frac{4}{5}$ lb. together. *Sum* 3*qr.* 14*lb.* 12*oz.* $12\frac{4}{5}$*dr.*

23. Add $\frac{1}{4}$ mile and $\frac{2}{5}$ furlong together. *Sum* 2*fur.* 16*p.*

24. Add $\frac{1}{4}$ of a square mile, $\frac{9}{10}$ of an acre, and $\frac{7}{8}$ of a rood together. *Sum* 161*a.* 19*p.*

25. Add $\frac{2}{9}$ barrel, $\frac{3}{8}$ gallon, and $\frac{3}{4}$ quart of beer together.
Sum 8gal. 2qt. $\frac{1}{2}$*pt.*

191. *When the fractions are such as will not reduce to known quantities without remainder.*

RULE. Reduce the given fractions to fractions of the highest denomination mentioned, (Art. 183.) then reduce these to a common denominator, (Art. 180.) add the numerators together, as in Art. 187, and reduce the sum to its proper quantity, (by Art. 186.) which will be the answer[a].

26. Add $\frac{3}{11}$ of a pound, $\frac{5}{7}$ shilling, and $\frac{8}{9}$ penny together.

OPERATION.

$$\textit{First, } \frac{5}{7}\textit{ shill.} = \frac{5}{7 \times 20} = \frac{5}{140} = \frac{1}{28}\textit{ L.}$$

$$\frac{8}{9}\textit{ penny} = \frac{8}{9 \times 12 \times 20} = \frac{8}{2160} = \frac{1}{270}\textit{ L.}$$

$$\left.\begin{array}{l} \textit{Then } 3 \times 28 \times 270 = 7560 \\ 1 \times 11 \times 270 = 2970 \\ 1 \times 11 \times 28 = 308 \end{array}\right\} \textit{new num.}$$

$$\overline{10838}\textit{ sum.}$$

$$11 \times 28 \times 270 = 83160\textit{ com. den.}$$

$$\textit{Wherefore } \frac{10838}{83160}\textit{ L.} = \frac{5419}{41580}\textit{ L.} = \frac{5419 \times 20}{41580} = 2s.\ 7d.\ \frac{193}{693}\textit{ the answer.}$$

Explanation.

I first reduce $\frac{5}{7}$*shill.* and $\frac{8}{9}$*d.* to fractions of a pound, which is the highest denomination in the question. I have then $\frac{3}{11}$, $\frac{1}{28}$, and $\frac{1}{270}$, all frac-

---

[a] This rule will be readily understood; for it is plain, from what has been shewn in the preceding notes, that in order to add fractions together, they must be reduced first to parts of the same whole, and then to parts of the same denomination; after which the sum is evidently found (as before) by adding all the new numerators together, placing the common denominator under the sum, and reducing this fraction to its equivalent value in known denominations.

tions of a pound, these I reduce to a common denominator, and add as in Art. 180; the sum $\frac{10838}{83160}$ is next reduced to its lowest terms, and lastly to its proper quantity, which is the answer.

27. Add $\frac{1}{7}$ lb. $\frac{1}{9}$ oz. and $\frac{1}{5}$ dwt. together.

Thus, $\frac{1}{9}oz. = \frac{1}{9 \times 12} = \frac{1}{108}lb.$ $\frac{1}{5}dwt. = \frac{1}{5 \times 20 \times 12}$ $= \frac{1}{1200}lb.$

$$\left.\begin{array}{l} \textit{Then } 1 \times 108 \times 1200 = 129600 \\ \quad\quad\; 1 \times 7 \times 1200 = \;\;\;8400 \\ \quad\quad\; 1 \times 7 \times 108 = \;\;\;\;\;756 \end{array}\right\} \textit{new num.}$$

$$138756 \textit{ their sum.}$$

$$7 \times 108 \times 1200 = 907200 \textit{ com. den.}$$

Wherefore $\frac{138756}{907200}lb. = \frac{11563}{75600}lb. = 1oz.\ 16dwt.\ 16gr.\ \frac{104}{105}$ *Ans.*

28. Add $\frac{1}{9}$ of a pound to $\frac{3}{5}$ of a shilling. *Sum* 2*s*. 9*d*. $\frac{13}{15}$.

29. Add $\frac{2}{3}$ of 5 pounds to $\frac{1}{10}$ of seven shillings. *Sum* 3*l*. 7*s*. 4*d*. $\frac{2}{5}$.

30. Add $\frac{1}{9}$ of a week, $\frac{1}{10}$ of a day, and $\frac{10}{11}$ of an hour together. *Sum* 21*h*. 55*m*. 12″.

SUBTRACTION OF VULGAR FRACTIONS.

192. *When the given fractions are both simple.*

RULE I. Reduce the fractions to a common denominator, beginning with the numerator of the greater fraction.

II. Subtract the lower new numerator from the upper, and under the remainder place the common denominator; this fraction being reduced to its lowest terms will be the answer[a].

---

[a] As fractions of different denominations cannot be added, so neither can they be subtracted; we must first reduce them to a common denominator, and then it is plain that their difference is found by taking the difference of the new numerators, and placing it over the common denominator. Thus in Ex. 1.

EXAMPLES.

1. From $\frac{8}{9}$ take $\frac{5}{7}$.

OPERATION.

Thus, $\left.\begin{array}{r} 8 \times 7 = 56 \\ 5 \times 9 = 45 \end{array}\right\}$ *new num.*

$\overline{11}$ *difference.*

$9 \times 7 = 63$ *common den.*

Answer $\frac{11}{63}$ *the difference required.*

Explanation.

Beginning with the numerator 8 of the greater fraction, I reduce the fractions to a common denominator; I then subtract 45 from 56, and place the remainder 11 over the common denominator 63, which fraction being in its lowest terms, is the answer.

2. From $\frac{11}{12}$ take $\frac{7}{8}$.

Thus, $\left.\begin{array}{r} 11 \times 8 = 88 \\ 7 \times 12 = 84 \end{array}\right\}$ *new num.*

$\overline{4}$ *the difference.*

$12 \times 8 = 96$ *common den.*

Wherefore $\frac{4}{96} = \frac{1}{24}$ *the answer.*

3. From $\frac{6}{7}$ take $\frac{3}{8}$, and from $\frac{1}{2}$ take $\frac{1}{3}$. *Diff.* $\frac{27}{56}$ *and* $\frac{1}{6}$.

4. From $\frac{2}{3}$ take $\frac{5}{9}$, and from $\frac{7}{8}$ take $\frac{3}{4}$. *Diff.* $\frac{1}{9}$ *and* $\frac{1}{8}$.

5. From $\frac{11}{12}$ take $\frac{2}{5}$, and from $\frac{5}{6}$ take $\frac{5}{7}$. *Diff.* $\frac{31}{60}$ *and* $\frac{5}{42}$.

193. *When there are mixed numbers, compound or complex fractions.*

RULE I. Reduce mixed numbers to improper fractions, compound and complex fractions to simple ones, reduce these to a common denominator, and proceed as before.

$\frac{8}{9}$ and $\frac{5}{7}$ reduced become $\frac{56}{63}$ and $\frac{45}{63}$ respectively, and the difference of these is evidently $\frac{56-45}{63}$, or $\frac{11}{63}$, as in the example.

Nothing can be plainer than this rule; for if one fraction be subtracted from another of the same denomination, the remainder is evidently a fraction of the same denomination with both; thus, if *two fifths* be taken from *three fifths*, the remainder will be *one fifth;* and the same of other fractions.

II. If the answer be an improper fraction, reduce it to its equivalent whole or mixed number[b].

6. From $3\frac{1}{4}$ take $\frac{3}{4}$ of $\frac{5}{6}$.

OPERATION.

First, $3\frac{1}{4} = \frac{3 \times 4 + 1}{4} = \frac{13}{4}$.

Secondly, $\frac{3}{4}$ *of* $\frac{5}{6} = \frac{15}{24}$.

Thirdly, $\left.\begin{array}{r} 13 \times 24 = 312 \\ 15 \times 4 = \underline{60} \end{array}\right\}$ *new numerators.*

252 *difference.*

$4 \times 24 = 96$ *common denominator.*

Fourthly, $\frac{252}{96} = \frac{21}{8} = 2\frac{5}{8}$ *the answer.*

Explanation.

First, I reduce the mixed number to an improper fraction $\frac{13}{4}$. Secondly, the compound fraction to a simple one $\frac{15}{24}$. Thirdly, I reduce these two to a common denominator, and subtract, which gives the fraction $\frac{252}{96}$. Fourthly, I reduce this fraction to its lowest terms $\frac{21}{8}$, and this to a mixed number.

7. From $\frac{3\frac{1}{4}}{5}$ take $\frac{2\frac{2}{3}}{5\frac{1}{2}}$.

OPERATION.

First, $\frac{3\frac{1}{4}}{5} = \frac{3 \times 4 + 1}{5 \times 4} = \frac{13}{20}$.

Secondly, $\frac{2\frac{2}{3}}{5\frac{1}{2}} = \frac{(2 \times 3 + 2) \times 2}{(5 \times 2 + 1) \times 3} = \frac{16}{33}$.

Thirdly, $\left.\begin{array}{r} 13 \times 33 = 429 \\ 16 \times 20 = \underline{320} \end{array}\right\}$ *new numerators.*

109 *difference.*

$20 \times 33 = 660$ *common denominator.*

Fifthly, $\frac{109}{660}$ *the answer.*

Explanation.

The two complex fractions are first reduced to the simple ones $\frac{13}{20}$ and $\frac{16}{33}$; these are next reduced to a common denominator; and all the rest as before.

---

[b] This rule is sufficiently plain from what is shewn in the note on the similar rule in Addition of Fractions, (Art. 189.)

8. From $\frac{2}{3}$ of $\frac{5}{6}$ take $\frac{4}{5}$ of $\frac{1}{3}$ of $\frac{2}{3}$. *Diff.* $\frac{17}{45}$.

9. From $2\frac{2}{3}$ take $1\frac{5}{6}$, and from $1\frac{1}{2}$ take $\frac{11}{12}$. *Diff.* $\frac{5}{6}$ *and* $\frac{7}{12}$.

10. From $\frac{2}{3\frac{1}{3}}$ take $\frac{3\frac{1}{3}}{7}$, and from $\frac{7}{9}$ take $\frac{1}{8}$ of $2\frac{1}{4}$. *Diff.* $\frac{13}{105}$ *and* $\frac{143}{288}$.

11. From $\frac{3}{5}$ of 4 take $\frac{2}{7}$ of 3. *Diff.* $1\frac{19}{35}$.

12. From $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ take $\frac{3\frac{2}{3}}{5\frac{4}{5}}$. *Diff.* $\frac{41}{348}$.

194. *To subtract mixed numbers, without reducing them to improper fractions.*

RULE I. Reduce the fractional parts to a common denominator, and having subtracted the less whole number from the greater, subjoin the difference of the fractions to the difference of the whole numbers for the answer.

II. But if the lower new numerator is greater than the upper, subtract it from the common denominator, add the remainder to the upper numerator, and set the sum over the common denominator for the fractional part; then carry 1 to the less whole number before you subtract it from the greater[c].

---

[c] The first part of the rule evidently supposes that the whole number and fraction to be subtracted is *each* less than the whole number and fraction from which they are respectively to be taken; now it is plain, that if $2\frac{3}{5}$ be taken from $3\frac{4}{5}$, the remainder is $1\frac{1}{5}$; and it will be equally so when applied to other similar examples.

But with respect to the second part of the rule, where the lower new numerator is the greater, subtracting it from the common denominator is equivalent to borrowing 1, (as in simple subtraction); thus in Ex. 14, where $6\frac{36}{45}$ is to be taken from $12\frac{10}{45}$, I say, 36 from 10 I cannot, I therefore borrow 45, (or $\frac{45}{45}$, which is equal to 1,) then 36 from 45, and 9 remain; this added to the 10 gives 19, viz. $\frac{19}{45}$; then, because I borrowed 1 (or $\frac{45}{45}$) in the fraction, I must carry 1 to the subtrahend of the whole numbers; wherefore carrying 1 to

13. From $4\frac{4}{5}$ take $1\frac{3}{7}$.

OPERATION.

First, $\left.\begin{array}{l} 4 \times 7 = 28 \\ 3 \times 5 = 15 \end{array}\right\}$ *new numerators.*

$\overline{13}$ *difference.*

$5 \times 7 = 35$ *common denominator.*

Then $4 - 1 = 3$ *difference of the whole numbers.*

Wherefore $3\frac{13}{35}$ *the answer.*

Explanation.

Having reduced the fractions $\frac{4}{5}$ and $\frac{3}{7}$ to a common denominator, and subtracted as before, I next subtract the whole numbers 1 from 4, and 3 remains; to this I subjoin the fraction $\frac{13}{35}$ for the answer.

14. From $12\frac{2}{9}$ take $6\frac{4}{5}$.

OPERATION.

First, $\left.\begin{array}{l} 2 \times 5 = 10 \\ 4 \times 9 = 36 \end{array}\right\}$ *new numerators.*

$9 \times 5 = 45$ *common denominator.*

Secondly, $45 - 36 + 10 = 19$.

Therefore the fraction is $\frac{19}{45}$.

Thirdly, carrying 1 *to the* $6 = 7$.

Therefore $12 - 7 = 5$ *the whole number.*

Whence $5\frac{19}{45}$ *the answer.*

Explanation.

Having first reduced the fractions $\frac{2}{9}$ and $\frac{4}{5}$ to a common denominator, secondly, I subtract 36 from 45, and add 10 to the remainder, which gives 19, this placed over the common denominator 45, gives $\frac{19}{45}$ for the fractional part; I then carry 1 to the whole number 6 is 7, this I subtract from the whole number 12, and 5 remains for the whole number; wherefore $5\frac{19}{45}$ is the answer required.

---

6 makes 7, which taken from 12 leaves 5; all this is evident, being exactly the process of simple subtraction.

15. From $18\frac{1}{2}$ take $4\frac{6}{7}$.

Thus, $\left.\begin{array}{l}1 \times 7 = 7 \\ 6 \times 2 = 12\end{array}\right\}$ *new numerators.*

$2 \times 7 = 14$ *common denominator.*

Then $14 - 12 + 7 = 9$, *therefore* $\frac{9}{14}$ *the fraction; also carry 1 to the* 4 *is* 5, *then* $18 - 5 = 13$ *the whole number; wherefore* $13\frac{9}{14}$ *the difference required.*

16. From $5\frac{2}{9}$ take $2\frac{4}{5}$, and from $3\frac{1}{3}$ take $2\frac{3}{5}$. *Diff.* $2\frac{19}{45}$ *and* $\frac{11}{15}$.

17. From $11\frac{2}{3}$ take $3\frac{1}{8}$, and from 11 take $9\frac{1}{2}$. *Diff.* $8\frac{13}{24}$ *and* $1\frac{1}{2}$.

18. From $20\frac{4}{5}$ take $4\frac{3}{4}$, and from $1\frac{1}{2}$ take $\frac{3}{4}$. *Diff.* $16\frac{1}{20}$ *and* $\frac{3}{4}$.

19. From $32\frac{1}{8}$ take $1\frac{1}{2}$, and from $7\frac{1}{6}$ take $4\frac{2}{3}$. *Diff.* $30\frac{5}{8}$ *and* $2\frac{1}{2}$.

195. *To subtract a proper fraction from a whole number.*

RULE I. Subtract the numerator of the fraction from the denominator, and place the remainder over the denominator for the fractional part.

2. Subtract 1 from the whole number, prefix the remainder to the fractional part, and it will give the answer[d].

---

[d] This rule depends on the same considerations with the preceding, as may be seen by an attentive examination of the 20th example, in which it is required to take $\frac{5}{9}$ from 8; now here we borrow 1, or $\frac{9}{9}$, from which taking $\frac{5}{9}$, the remainder is $\frac{4}{9}$; then, because we borrowed ($\frac{9}{9}$, or) 1, we must compensate by (subtracting or) lessening the 8 by the 1 we borrowed.

20. From 8 take $\frac{5}{9}$.

Thus, $9 - 5 = 4$, *wherefore* $\frac{4}{9}$ *is the fraction; then* $8 - 1 = 7$ *the whole number; wherefore* $7\frac{4}{9}$ *the answer required.*

21. From 10 take $\frac{5}{12}$.

Thus, $12 - 5 = 7$, *wherefore* $\frac{7}{12}$ *the fraction; also* $10 - 1 = 9$ *the whole number; wherefore* $9\frac{7}{12}$ *the answer.*

22. From 4 take $\frac{2}{3}$, and from 1 take $\frac{8}{9}$. *Diff.* $3\frac{1}{3}$ *and* $\frac{1}{9}$.

23. From 25 take $\frac{5}{7}$, and from 10 take $\frac{7}{9}$. *Diff.* $24\frac{2}{7}$ *and* $9\frac{2}{9}$.

24. From 1 take $\frac{3}{5}$, and from 2 take $\frac{2}{7}$. *Diff.* $\frac{2}{5}$ *and* $1\frac{5}{7}$.

196. *When the fractions are of different denominations in money, weights, or measures.*

RULE. Reduce the fractions to their proper quantities, then subtract by the rules of Compound Subtraction *.

25. From $\frac{3}{8}$ of a pound take $\frac{5}{6}$ of a shilling.

OPERATION.

$$\begin{array}{llllr} & & & s. & d. \\ \text{Thus, } \frac{3}{8}\,L. = \frac{3 \times 20}{8} & = \frac{60}{8} & = & 7 & 6 \\ \frac{5}{6}\,s. \quad = \frac{5 \times 12}{6} & = \frac{60}{6} & = & 0 & 10 \\ & & \textit{Answer} & 6 & 8 \end{array}$$

Explanation.

Having reduced the fractions to their proper quantities, I subtract the lower 10*d.* from the upper 7*s.* 6*d.*, and the remainder 6*s.* 8*d.* is the answer.

* The grounds of this rule are explained in the note on the similar rule in Addition of Fractions, Art. 190.

26. From $\frac{4}{5}$ of a lb. troy take $\frac{7}{8}$ of an ounce.

$$\begin{array}{llll} & oz. & dwt. & gr. \\ \textit{Thus, } \frac{4}{5}\,lb. = \frac{4 \times 12}{5} = \frac{48}{5} = & 9 & 12 & 0 \\ \frac{7}{8}\,oz. = \frac{7 \times 20}{8} = \frac{140}{8} = & 0 & 17 & 12 \\ \hline & 8 & 14 & 12 \ \textit{Answer.} \end{array}$$

27. From $\frac{3}{4}$ of a pound take $\frac{1}{6}$ of a shilling. *Diff.* 14*s*. 10*d*

28. From $\frac{2}{3}$ of a guinea take $\frac{7}{12}$ of a pound. *Diff.* 2*s*. 4*d*.

29. From $\frac{5}{6}$ of a pound take $\frac{11}{40}$ of a crown. *Diff.* 15*s*. 3*d*.$\frac{1}{2}$.

30. From $\frac{3}{5}$ of a ton take $\frac{7}{8}$ of a cwt. *Diff.* 11*cwt*. 0*qr*. 14*lb*.

31. From $\frac{31}{80}$ of a mile take $\frac{1}{4}$ of a furlong. *Diff.* 2*fur*. 34*p*.

32. From $3\frac{3}{7}$ weeks take $4\frac{5}{8}$ hours. *Diff.* 3*w*. 2*d*. 23*h*. ?
22*m*. 30″.

197. *When the fractions will not reduce to known quantities without remainder.*

RULE. Reduce the given fractions to fractions of the greatest denomination mentioned, reduce the latter to a common denominator, subtract the less numerator from the greater, as in Art. 192, then having placed the remainder over the common denominator, reduce this fraction to its proper quantity, by Art. 186[f].

[f] This rule depends on the same principles with the similar rule in Addition of Fractions, Art. 191. It will, in some cases, be more convenient to reduce the fraction of the greater denomination to an equivalent one of the less, and proceed according to the latter part of the rule.

33. From $\frac{3}{7}$ of a pound take $\frac{5}{9}$ of a shilling.

OPERATION.

Thus, $\frac{5}{9}s. = \frac{5}{9 \times 20} = \frac{5}{180} = \frac{1}{36}L.$

Then $\left.\begin{array}{r} 3 \times 36 = 108 \\ 1 \times 7 = 7 \end{array}\right\}$ *new numerators.*

$\overline{101}$ *difference.*

$7 \times 36 = 252$ *common denominator.*

Therefore $\frac{101}{252}L. = \frac{101 \times 20}{252} = \frac{2020}{252} = \frac{505}{63} = 8\frac{1}{63}$ *shillings, the answer.*

Explanation.

First, $\frac{5}{9}$ of a shilling reduced to the fraction of a pound is $\frac{1}{36}$; then I reduce $\frac{3}{7}$ and $\frac{1}{36}$ to a common denominator; then subtracting 7 from 108, the remainder is 101. Lastly, I reduce $\frac{101}{252}$ *L.* to its proper quantity, which is $8\frac{1}{63}$ *shill.*

34. From $\frac{3}{11}$ of a hhd. of wine take $3\frac{1}{5}$ gallons.

Thus, $3\frac{1}{5}$ *gal.* $= \frac{16}{5}$ *gal.* $= \frac{16}{5 \times 63} = \frac{16}{315}$ *hhd.*

Then $\left.\begin{array}{r} 3 \times 315 = 945 \\ 16 \times 11 = 176 \end{array}\right\}$ *new numerators.*

$\overline{769}$ *difference.*

$11 \times 315 = 3465$ *common denominator.*

Wherefore $\frac{769}{3465}$ *hhd.* $= \frac{769 \times 63}{3465} = \frac{48447}{3468} = \frac{16149}{1156} =$

13*gal.* 3*qt.* 1*pt.* $3\frac{9}{289}$ *gills, the answer.*

25. From $\frac{5}{7}$ of a pound take $\frac{4}{5}$ of a shilling. *Diff.* 13*s.* 5*d.* $\frac{29}{35}$.

36. From $8\frac{5}{12}$ acres take $7\frac{3}{4}$ poles. *Diff.* 8*a.* 1*r.* 18*p.* $\frac{11}{12}$.

37. From $\frac{2}{11}$ of a barrel take $\frac{8}{9}$ of a gallon. *Diff.* 5*gal.* 2*qt.* 1*pt.*$\frac{75}{297}$.

38. From $\frac{3}{5}$ of a chaldron take $\frac{5}{9}$ of a sack. *Diff.* 6*sa.* 1*b.* 3*pk.*$\frac{11}{15}$.

MULTIPLICATION OF VULGAR FRACTIONS.

198. *To multiply fractions together.*

RULE I. Multiply the numerators together for a numerator, and the denominators together for a denominator.

II. If the new fraction be a proper fraction, reduce it to its lowest terms; if an improper one, reduce it to its equivalent whole or mixed number[a].

EXAMPLES.

1. Multiply $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{7}{8}$ together.

OPERATION.

Thus, $\frac{3}{4} \times \frac{2}{3} \times \frac{7}{8} = \frac{42}{96} = \frac{7}{16}$ *answer.*

Explanation.

I first multiply the numerators 3, 2, and 7 together, and the product is 42.

---

[a] To multiply a fraction by a whole number, we must evidently multiply the numerator by the whole number; but to divide it, we must multiply the denominator by the whole number: thus, if $\frac{1}{4}$ be multiplied by 2, the product will be $\frac{2}{4}$ (or $\frac{1 \times 2}{4}$). But $\frac{1}{4}$ divided by 2 is evidently $\frac{1}{8}$, (or $\frac{1}{4 \times 2}$) for two fourths are double of one fourth, and one eighth is the half of one fourth: this being premised, let it be required to multiply $\frac{2}{3}$ by $\frac{4}{5}$; now 2 multiplied by 4 equals 8; but it is not 2, but a third part of 2, which is to be multiplied, and therefore the product will be the third part of 8 only, or $\frac{8}{3}$; but the multiplier is not 4, but the fifth part of 4 only, wherefore the product will be the fifth part only of $\frac{8}{3}$, that is, $\frac{8}{15}$; wherefore $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$; or the numerators multiplied together give the numerators of the product, and the denominators multiplied give the denominators; which is the rule.

Then I multiply the denominators 4, 3, and 8 together, and the product is 96. I then reduce $\frac{42}{96}$ to its lowest terms $\frac{7}{16}$.

2. Multiply $\frac{4}{5}$ and $\frac{11}{3}$ together.

Thus, $\frac{4}{5} \times \frac{11}{3} = \frac{44}{15} = 2\frac{14}{15}$ *the product.*

3. Multiply $\frac{2}{3}$ and $\frac{4}{5}$ of $\frac{6}{7}$ together.

Thus, $\frac{2}{3} \times \frac{4}{5}$ *of* $\frac{6}{7} = \frac{48}{105} = \frac{16}{35}$ *the product.*

4. Multiply $\frac{4}{9}$ and $\frac{2}{5}$ together. *Prod.* $\frac{8}{45}$.

5. Multiply $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{12}{13}$ continually together. *Product* $\frac{45}{104}$.

6. Multiply $\frac{2}{3}$ of $\frac{4}{5}$ and $\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{9}{10}$ together. *Product* $\frac{24}{175}$.

7. Multiply $\frac{12}{5}$, $\frac{7}{8}$, and $\frac{3}{4}$ of $\frac{6}{7}$ together. *Prod.* $1\frac{7}{20}$.

199. ***When mixed numbers or complex fractions are to be multiplied.***

RULE. Reduce the mixed numbers to improper fractions, and the complex fractions to simple ones, and proceed as before[b].

8. Multiply $3\frac{1}{2}$ by $\frac{4\frac{1}{3}}{5}$.

OPERATION.

First, $3\frac{1}{2} = \frac{3 \times 2 + 1}{2} = \frac{7}{2}$.

Secondly, $\frac{4\frac{1}{3}}{5} = \frac{4 \times 3 + 1}{5 \times 3} = \frac{13}{15}$.

Thirdly, $\frac{7}{2} \times \frac{13}{15} = \frac{91}{30} = 3\frac{1}{30}$ *the product.*

[b] The reason of this process evidently follows from the preceding note.

Explanation.

First, I reduce $3\frac{1}{2}$ to the improper fraction $\frac{7}{2}$. Secondly, I reduce the complex fraction to its equal $\frac{13}{15}$. Thirdly, I multiply the two fractions $\frac{7}{2}$ and $\frac{13}{15}$ together, and reduce the product $\frac{91}{30}$ to its equivalent mixed number $3\frac{1}{30}$.

9. Multiply $\frac{4}{2\frac{1}{4}}$, $\frac{3\frac{4}{7}}{8}$, and $\frac{2\frac{2}{3}}{3\frac{3}{5}}$ together.

Thus, $\frac{4}{2\frac{1}{4}} = \frac{4 \times 4}{2 \times 4 + 1} = \frac{16}{9}$; $\frac{3\frac{4}{7}}{8} = \frac{3 \times 7 + 4}{8 \times 7} = \frac{25}{56}$;

$\frac{2\frac{2}{3}}{3\frac{3}{5}} = \frac{(2 \times 3 + 2) \times 5}{(3 \times 5 + 3) \times 3} = \frac{40}{54} = \frac{20}{27}$;

Then $\frac{16}{9} \times \frac{25}{56} \times \frac{20}{27} = \frac{8000}{13608} = \frac{1000}{1701}$ *the product required.*

10. Multiply $2\frac{2}{3}$ by $3\frac{3}{4}$. *Prod.* 10.

11. Multiply $\frac{5}{9}$, $4\frac{5}{6}$, and $1\frac{1}{2}$ together. *Prod.* $4\frac{1}{36}$.

12. Multiply $\frac{3}{4}$ of $2\frac{1}{2}$ and $\frac{5\frac{6}{7}}{8}$ together. *Prod.* $\frac{615}{448}$.

13. Multiply $\frac{1^{1}}{2\frac{1}{3}}$ and $\frac{2\frac{2}{7}}{3\frac{3}{4}}$ together. *Prod.* $\frac{128}{525}$.

200. *To multiply a whole number and fraction together.*

RULE. Multiply the numerator of the fraction by the whole number, and under the product set the denominator; then reduce this fraction to its lowest or proper terms, as the case requires, and it will be the answer[e].

---

[e] Whatever parts a fraction consists of, its product when multiplied by any whole number will consist of like parts; thus *two sevenths* multiplied by *three* will produce *six sevenths*, that is, $\frac{2}{7} \times 3 = \frac{2 \times 3}{7}$ $(= \frac{6}{7})$, which is the rule.

14. Multiply 2 by $\frac{3}{8}$.

OPERATION.

$2 \times \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$ *product.*

Explanation.

Here I multiply the numerator 3 by the whole number 2, and under the product 6 I place the denominator 8; then I reduce the fraction $\frac{6}{8}$ to its lowest terms $\frac{3}{4}$.

15. Multiply 2, $\frac{7}{9}$, $\frac{3}{8}$, and 4 continually together.

Thus, $2 \times \frac{7}{9} \times \frac{3}{8} \times 4 = \frac{168}{72} = \frac{7}{3} = 2\frac{1}{3}$ *the product.*

16. Multiply $\frac{3}{7}$, $\frac{8}{9}$, and 12 together. *Prod.* $4\frac{4}{7}$.

17. Multiply 7 by $\frac{6}{7}$ of $\frac{8}{9}$. *Prod.* $5\frac{1}{3}$.

18. Multiply $\frac{4}{7}$ of $\frac{1}{3}$ of $\frac{2}{9}$ by 3. *Prod.* $\frac{8}{63}$.

201. *When the whole number will divide the denominator of the fraction without remainder,* divide by it, and set the quotient under the given numerator; this fraction reduced as before will give the answer[d].

19. Multiply $\frac{1}{48}$ by 12.

OPERATION.

$\frac{1}{48} \times 12 = \frac{4}{48 \div 12} = \frac{1}{4}$ *prod.*

Explanation.

Here I divide the denominator 48 by the whole number 12, and set the quotient 4 under the numerator 1 for the answer.

[d] To divide the denominator by any number is the same as to multiply the numerator by it; for let $\frac{1}{4}$ be multiplied by 2, the product by the last rule will be $\frac{2}{4}$, which reduced to its lowest terms is $\frac{1}{2}$. Let now the denominator 4 of the fraction $\frac{1}{4}$ be divided by 2, thus, $\frac{1}{4 \div 2}$, and the quotient will be 2, making the resulting fraction $\frac{1}{2}$, the same as by the former method; and the same holds true in every other instance: wherefore the truth of the rule is manifest.

20. Multiply $\frac{119}{144}$ by 9.

Thus, $\frac{119}{144 \div 9} = \frac{119}{16} = 7\frac{7}{16}$ *the product.*

21. Multiply $\frac{2}{75}$ by 25. *Prod.* $\frac{2}{3}$.

22. Multiply $\frac{4}{45}$ by 15. *Prod.* $1\frac{1}{3}$.

23. Multiply $\frac{107}{108}$ by 54. *Prod.* $53\frac{1}{2}$.

202. *When a numerator and denominator are the same,* both may be omitted in the operation, a small line being drawn through the figures omitted; this operation is called cancelling[*].

24. Multiply $\frac{3}{4}$ and $\frac{4}{7}$ together.

Thus, $\frac{3}{\cancel{4}} \times \frac{\cancel{4}}{7} = \frac{3}{7}$ *the product. Here we left out the* 4, *which is both a numerator and a denominator.*

25. Multiply $\frac{1}{6}$, $\frac{5}{8}$, $\frac{6}{7}$, and $\frac{3}{5}$ together.

Thus, $\frac{1}{\cancel{6}} \times \frac{\cancel{5}}{8} \times \frac{\cancel{6}}{7} \times \frac{3}{\cancel{5}} = \frac{3}{56}$ *the product. Here we omit the* 6 *and* 6, *and the* 5 *and* 5.

---

* The numerator of a fraction may be considered as a *multiplier,* and the denominator as a *divisor;* it is plain, that if any number be both multiplied and divided by any (the same) number, the result will be the same as the given number; wherefore such multiplication and division (as they mutually destroy the effect of each other) may be omitted: which is what the rule directs.

It may be further observed, that one numerator cancels only one *equal* denominator, and *vice versa;* but it will cancel two or more when it is equal to their product: thus, in ex. 26. the numerator 4 cancels the denominator 4 only, but the denominator 6 cancels both the numerators 2 and 3, because it equals 2 × 3.

26. Multiply $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{6}{7}$ together.

Here the threes, fours, fives, and sixes being omitted, the product will be $\frac{2}{7}$.

27. Multiply $\frac{4}{7}$, $\frac{3}{5}$, $\frac{3}{4}$, and $\frac{7}{8}$ together. *Prod.* $\frac{9}{40}$.

28. Multiply $\frac{7}{8}$, $\frac{4}{5}$, $\frac{3}{4}$, and $\frac{8}{9}$ together. *Prod.* $\frac{7}{15}$.

203. ***When a numerator and a denominator can be divided by any (the same) number,*** **draw a small line through the numbers, and use the quotients instead of them**[f].

29. Multiply $\frac{8}{9}$ by $\frac{12}{36}$.

OPERATION.

Thus, $\frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{9}}} \times \frac{\overset{4}{\cancel{12}}}{\underset{9}{\cancel{36}}} = \frac{8}{27}$ *prod.*

Explanation.

Here 8 and 36 both divide by 4; I dash out these numbers, and write the quotients 2 and 9 opposite its respective number: also 12 and 9 divide by 3; I cancel these, and write down the quotients 4 and 3; then I multiply 2 by 4 and 3 by 9 for the answer.

30. Multiply $\frac{12}{13}$, $\frac{26}{27}$, $\frac{11}{18}$, and $\frac{6}{7}$ together.

OPERATION.

Thus, $\frac{\overset{2}{\cancel{12}}}{\cancel{13}} \times \frac{\overset{2}{\cancel{26}}}{27} \times \frac{11}{\underset{3}{\cancel{18}}} \times \frac{\overset{2}{\cancel{6}}}{7} = \frac{88}{189}$ *product.*

Explanation.

Here 12 and 18 are divisible by 6, which goes twice in the former and 3 times in the latter; I therefore put 2 opposite 12, and 3 opposite 18; 12 and 18 are therefore cancelled: in like manner 13 cancels 26 by 2, and 3 cancels 6 by 2; the *ones* need not be put down; I therefore multiply 2, 2, 11, and 2 together for the numerator, and 27 and 7 for the denominator.

---

[f] The truth of this will appear by multiplying the given fractions together, and reducing the product to its lowest terms; thus, ex. 29. $\frac{8}{9} \times \frac{12}{36} = \frac{96}{324}$, which reduced to its lowest terms is $\frac{8}{27}$, the same with the answer obtained by the rule. This method of cancelling, when it can be applied, saves much trouble.

31. Multiply $\frac{7}{8}$, $\frac{12}{13}$, $\frac{9}{14}$, $\frac{2}{14}$, and $\frac{1}{3}$ together.

$$Thus,\ \frac{\cancel{7}}{\underset{2}{\cancel{8}}} \times \frac{\overset{3}{\cancel{12}}}{13} \times \frac{\overset{3}{\cancel{9}}}{\underset{\cancel{7}}{\cancel{14}}} \times \frac{\cancel{2}}{14} \times \frac{1}{\cancel{3}} = \frac{9}{364}\ product.$$

32. Multiply $\frac{4}{5}$, $\frac{1}{2}$, and $\frac{10}{11}$ together. *Prod.* $\frac{4}{11}$.

33. Multiply $\frac{14}{15}$, $\frac{4}{21}$, $\frac{15}{16}$, and $\frac{8}{9}$ together. *Prod.* $\frac{4}{27}$.

34. Multiply $4\frac{1}{2}$, $2\frac{3}{4}$, and $\frac{4}{7}$ together. *Prod.* $7\frac{1}{14}$.

35. Multiply $2\frac{4}{5}$, $\frac{3}{7}$ of $\frac{1}{2}$, and $\frac{5}{6}$ together. *Prod.* $\frac{1}{2}$.

36. Multiply $\frac{2}{3}$ of $\frac{4}{5}$, $3\frac{7}{30}$, and $\frac{6}{7}$ together. *Prod.* $1\frac{251}{525}$.

DIVISION OF VULGAR FRACTIONS.

204. *To divide one fraction by another.*

RULE. Invert the divisor, and then multiply the fractions (with the divisor so inverted) together, as in Art. 198.

Mixed numbers must be reduced to improper fractions, and complex fractions to simple ones, previous to the operation[s].

[s] The reason of this rule will appear from the first example, where $\frac{3}{4}$ is required to be divided by $\frac{5}{7}$. First, suppose $\frac{3}{4}$ were to be divided by 5 only, it is evident that the quotient would be $\frac{3}{4 \times 5}$, or $\frac{3}{20}$; but instead of 5, the divisor $\frac{5}{7}$ is only one seventh part of 5, and therefore the quotient $\frac{3}{20}$, arising from a divisor seven times too great, must be seven times less than it ought to be, and consequently must be multiplied by 7 to make it right; wherefore $\frac{3 \times 7}{20} = \frac{21}{20}$ (as in the example) is the true quotient of $\frac{3}{4}$ divided by $\frac{5}{7}$; and this quotient arises by multiplying 3 by 7, and 4 by 5, or by inverting the divisor, as has been shewn.

EXAMPLES.

1. Divide $\frac{3}{4}$ by $\frac{5}{7}$.

OPERATION.

The divisor inverted is $\frac{7}{5}$.

Then $\frac{3}{4} \times \frac{7}{5} = \frac{21}{20} = 1\frac{1}{20}$ *quot.*

Explanation.

I first invert the divisor $\frac{5}{7}$, and it becomes $\frac{7}{5}$; I then multiply this and $\frac{3}{4}$ together, and reduce the product $\frac{21}{20}$ to a mixed number.

2. Divide $2\frac{1}{3}$ by $3\frac{1}{4}$.

OPERATION.

First, $2\frac{1}{3} = \frac{2 \times 3 + 1}{3} = \frac{7}{3}$.

$3\frac{1}{4} = \frac{3 \times 4 + 1}{4} = \frac{13}{4}$.

The divisor inverted is $\frac{4}{13}$.

Then $\frac{7}{3} \times \frac{4}{13} = \frac{28}{39}$ *quotient.*

Explanation.

I first reduce the two mixed numbers to the improper fractions $\frac{7}{3}$ and $\frac{13}{4}$; the latter, being the divisor, I invert, and multiply the former by it, which gives the quotient.

3. Divide $\frac{2\frac{5}{6}}{7}$ by $\frac{7}{8}$.

Thus, $\frac{2\frac{5}{6}}{7} = \frac{2 \times 6 + 5}{7 \times 6} = \frac{17}{42}$. *Then* $\frac{7}{8}$ *inverted is* $\frac{8}{7}$.

Therefore $\frac{17}{42} \times \frac{8}{7} = \frac{136}{294} = \frac{68}{147}$ *the quotient.*

4. Divide $\frac{4}{5}$ of $\frac{5}{6}$ by $\frac{3}{4}$ of $\frac{7}{10}$ of $\frac{2}{3}$.

The divisor inverted is $\frac{4}{3}$ *of* $\frac{10}{7}$ *of* $\frac{3}{2}$.

Then $\frac{4}{\cancel{5}}$ *of* $\frac{\cancel{5}}{\underset{3}{\cancel{6}}} \times \frac{\overset{2}{\cancel{4}}}{\cancel{3}}$ *of* $\frac{\overset{5}{\cancel{10}}}{7}$ *of* $\frac{\cancel{3}}{\cancel{2}} = \frac{40}{21} = 1\frac{19}{21}$ *the quotient.*

5. Divide $\frac{4}{7}$ by $\frac{2\frac{3}{5}}{4\frac{5}{7}}$.

The divisor reduced and inverted is $\frac{165}{91}$.

Therefore $\frac{4}{7} \times \frac{165}{91} = \frac{660}{637} = 1\frac{23}{637}$ *the quotient.*

6. Divide $\frac{7}{9}$ by $\frac{10}{11}$, and $\frac{5}{6}$ by $\frac{3}{5}$. *Quot.* $\frac{77}{90}$ *and* $1\frac{7}{18}$.

7. Divide $2\frac{6}{7}$ by $\frac{3}{5}$, and $\frac{1}{2}$ by $\frac{2}{3}$. *Quot.* $4\frac{16}{21}$ *and* $\frac{3}{4}$.

8. Divide $\frac{4}{5}$ of $\frac{2}{3}$ by $9\frac{9}{10}$, and $2\frac{1}{4}$ by $\frac{9}{10}$ *Quot.* $\frac{16}{297}$ *and* $2\frac{1}{2}$.

9. Divide $\frac{11}{12}$ by $\frac{3}{8}$ of $\frac{2}{3}$, and $\frac{3}{7}$ by $\frac{9}{13}$. *Quot.* $3\frac{2}{3}$ *and* $\frac{13}{21}$.

10. Divide $\frac{7\frac{7}{8}}{14}$ by $12\frac{3}{7}$, and $4\frac{1}{2}$ by $5\frac{3}{4}$. *Quot.* $\frac{21}{464}$ *and* $\frac{18}{23}$.

11. Divide $8\frac{9}{10}$ by $\frac{7}{8\frac{3}{4}}$, and $\frac{3}{7}$ by $1\frac{2}{3}$. *Quot.* $11\frac{1}{8}$ *and* $\frac{9}{35}$.

12. Divide $\frac{1\frac{1}{2}}{2\frac{2}{3}}$ by $\frac{1}{2}$ of $\frac{2}{5}$ of $\frac{7}{8}$. *Quot.* $3\frac{3}{14}$.

205. *When the numerator of the divisor will divide the numerator of the dividend, and the denominator divide the denominator, both without remainder*; then (without inverting the divisor) divide numerator by numerator, and denominator by denominator, and the quotients will form a fraction, which (reduced if necessary) will be the answer[h].

13. Divide $\frac{12}{35}$ by $\frac{4}{5}$.

OPERATION.	*Explanation.*
Thus, $\frac{12}{35} \div \frac{4}{5} = \frac{3}{7}$ *quotient.*	I put the sign of division between the fractions, then divide 12 by 4, and 35 by 5.

[h] The truth of this process will appear by working the examples included under this rule by the former rule, as in both cases the same result will be produced; and it is recommended to prove every operation in this by the former rule.

14. Divide $\frac{120}{121}$ by $\frac{3}{11}$.

Thus, $\frac{120}{121} \div \frac{3}{11} = \frac{40}{11} = 3\frac{7}{40}$ *quotient.*

15. Divide $12\frac{9}{20}$ by $1\frac{1}{2}$.

Thus, $12\frac{9}{20} = \frac{249}{20}$, *and* $1\frac{1}{2} = \frac{3}{2}$; *then* $\frac{249}{20} \div \frac{3}{2} = \frac{83}{10}$ $= 8\frac{3}{10}$ *the quotient.*

16. Divide $\frac{8}{9}$ by $\frac{1}{3}$, and $\frac{1}{12}$ by $\frac{1}{3}$. *Quot.* $2\frac{2}{3}$ *and* $\frac{1}{4}$.

17. Divide $\frac{12}{35}$ by $\frac{6}{7}$, and $\frac{20}{21}$ by $\frac{4}{7}$. *Quot.* $\frac{2}{5}$ *and* $1\frac{2}{3}$.

18. Divide $3\frac{9}{40}$ by $\frac{3}{5}$, and $4\frac{2}{9}$ by $\frac{2}{3}$. *Quot.* $5\frac{3}{8}$ *and* $6\frac{1}{3}$.

19. Divide $25\frac{3}{25}$ by $\frac{1}{5}$ of 4. *Quot.* $31\frac{2}{5}$.

206. *To divide a whole number by a fraction.*

Rule. Invert the fraction, and multiply the numerator of the inverted fraction by the whole number, under the product place the denominator, and reduce this fraction (if necessary) for the answer[1].

20. Divide 12 by $\frac{7}{8}$.

The divisor inverted is $\frac{8}{7}$; *then* $12 \times \frac{8}{7} = \frac{96}{7} = 13\frac{5}{7}$ *the quotient required.*

21. Divide 20 by $\frac{2}{3}$.

Thus, $20 \times \frac{3}{2} = \frac{60}{2} = 30$ *the quotient.*

22. Divide 4 by $\frac{5}{8}$, and 2 by $\frac{1}{3}$. *Quot.* $6\frac{2}{5}$ *and* 6.

---

[1] If 1 be put for a denominator to the whole number, it will become a fraction, and the divisor being inverted, this rule will coincide with the first rule in Division of Fractions, and is consequently founded on the same principles.

23. Divide 9 by $\frac{2}{3}$, and 10 by $\frac{4}{7}$. *Quot.* $13\frac{1}{2}$ *and* $17\frac{2}{7}$.

24. Divide 1 by $\frac{1}{2}$, and 3 by $\frac{9}{13}$. *Quot.* 2 *and* $4\frac{3}{13}$.

25. Divide 9 by $\frac{9}{10}$, and 11 by $\frac{2}{5}$. *Quot.* 10 *and* $27\frac{1}{2}$.

207. *To divide a fraction by a whole number.*

RULE. Multiply the denominator of the fraction by the whole number, and over it set the numerator[k].

26. Divide $\frac{3}{4}$ by 5.

Thus, $\frac{3}{4 \times 5} = \frac{3}{20}$ *the quotient.*

27. Divide $\frac{7}{8}$ by 14.

Thus, $\frac{7}{8 \times 14} = \frac{7}{112} = \frac{1}{16}$ *the quotient.*

28. Divide $\frac{4}{5}$ by 3, and $\frac{1}{3}$ by 4. *Quot.* $\frac{4}{15}$ *and* $\frac{1}{12}$.

29. Divide $\frac{11}{12}$ by 10, and $\frac{40}{5}$ by 4. *Quot.* $\frac{11}{120}$ *and* 2.

208. *When the whole number will divide the numerator of the fraction without remainder,* then divide the numerator by it, and under the quotient set the denominator[l].

30. Divide $\frac{12}{13}$ by 4.

Thus, $\frac{12 \div 4}{13} = \frac{3}{13}$ *the quotient.*

---

[k] Place 1 as a denominator to the whole number, and this rule will coincide with the first, in the same manner as the preceding rule has been shewn to coincide with it.

[l] The truth of this rule is evident from ex. 30; for the one fourth part of *twelve* thirteenths is evidently *three* thirteenths: and the like will appear from other examples.

31. Divide $3\frac{4}{7}$ by 5.

Thus, $3\frac{4}{7} = \frac{25}{7}$; *then* $\frac{25 \div 5}{7} = \frac{5}{7}$ *the quotient.*

32. Divide $\frac{5\frac{2}{3}}{6\frac{1}{4}}$ by 34.

Thus, $\frac{5\frac{2}{3}}{6\frac{1}{4}} = \frac{68}{75}$; *then* $\frac{68 \div 34}{75} = \frac{2}{75}$ *the quotient.*

33. Divide $\frac{10}{11}$ by 5, and $\frac{21}{23}$ by 7. *Quot.* $\frac{2}{11}$ *and* $\frac{3}{23}$.

34. Divide $6\frac{7}{8}$ by 11, and $\frac{2}{3}$ of $7\frac{1}{9}$ by 8. *Quot.* $\frac{5}{8}$ *and* $\frac{16}{27}$.

35. Divide $\frac{2}{3}$ of $\frac{4}{5}$ of $5\frac{1}{2}$ by 2. *Quot.* $1\frac{7}{15}$.

209. Promiscuous Examples for Practice.

1. What is the sum and the difference of $\frac{4}{7}$ and $\frac{11}{13}$? *Answer, sum* $1\frac{38}{91}$, *diff.* $\frac{25}{91}$.

2. Required the product and quotient of $3\frac{7}{8}$, by $\frac{7}{8}$ of 3. *Answer, prod.* $10\frac{11}{64}$, *quot.* $1\frac{10}{21}$.

3. Which is greatest, the sum of $\frac{5}{7}$ and $\frac{11}{12}$, or the difference of $1\frac{7}{10}$ and $\frac{29}{30}$, and how much? *Answer, the sum, by* $\frac{377}{420}$.

4. Which is the greater, the product of $\frac{2}{5}$ by $\frac{7}{9}$, or their quotient, and how much? *Answer, the quotient, by* $\frac{64}{315}$.

5. How much is $\frac{10}{9}$ greater than $\frac{9}{10}$? *Answer* $\frac{19}{90}$.

6. Which is greatest, the quotient of $\frac{3}{4}$ by $\frac{5}{6}$, or that of $\frac{5}{6}$ by $\frac{3}{4}$, and how much? *Answer, the latter, by* $\frac{19}{90}$.

7. If $\frac{9}{10}$ be multiplied and divided by $\frac{7}{8}$, which is the greatest; the product or the quotient, and how much? *Answer, the quotient, by* $\frac{27}{112}$.

8. If $2\frac{7}{8}$ be added to $\frac{4}{5}$, and also divided by it, which is greatest, the sum or the quotient, and how much? *Answer, the sum, by* $\frac{13}{160}$.

9. What sum will arise by adding the sum and difference of $\frac{4}{5}$ of 3 guineas and $\frac{2}{3}$ of 4 pounds together? *Ans.* 5*l.* 6*s.* 8*d.*

10. If the quotient of $2\frac{3}{4}$ by $\frac{4}{5}$ of 2 be multiplied by the sum, what is the product? *Answer* $7\frac{61}{128}$.

210. PROPORTION, OR, THE RULE OF THREE IN VULGAR FRACTIONS.

RULE I. Examine the question so as to be able to determine how the stating is to be made, then reduce the first and third terms to fractions of the same denomination, if they are not so already, and the second to a fraction of the greatest denomination contained in it, or of a greater denomination if convenient.

II. With the fractions to which the given numbers are reduced state the question, and examine whether the answer will be greater or less than the second term; if *greater*, mark the *less* extreme for a divisor, but if *less*, mark the *greater*.

III. Invert the marked term, and then multiply the three terms continually together; the product will be a fraction of the same denomination with that which the second term was reduced to, and must be reduced to its proper quantity for the answer*.

---

* This rule is founded on the same principles with the Rule of Three in whole numbers, (Art. 126,) and under it are included both the direct and inverse rules, which in effect are only branches of one and the same general rule.

EXAMPLES.

1. If $2\frac{1}{5}$*cwt.* of cheese cost 10*l.* 2*s.* 6*d.* what cost 1*cwt.* 1*qr.* 14*lb.*?

Reduction of the terms.

$$2\frac{1}{5}cwt. = \frac{11}{5}cwt. \qquad 10l.\ 2s.\ 6d. = 10\frac{1}{8}L. = \frac{81}{8}L.$$

$$1cwt.\ 1qr.\ 14lb. = 1\frac{3}{8}cwt. = \frac{11}{8}cwt.$$

Stating.

$$*\frac{11}{5}cwt. : \frac{81}{8}L. :: \frac{11}{8}cwt. :$$

OPERATION.

$$\frac{81}{8} \times \frac{11}{8} \times \frac{5}{11} = \frac{405}{64}L. = 6l.\ 6s.\ 6d.\tfrac{3}{4}\ answer.$$

Explanation.

I first reduce the terms which will be the first and third to fractions of an hundred weight, viz. $2\frac{1}{5}$ to $\frac{11}{5}$*cwt.*, and 1*cwt.* 1*qr.* 14*lb.* to $\frac{11}{8}$: next I reduce 10*l.* 2*s.* 6*d.*, the second, to $\frac{81}{8}$*L.*: I then state the question, from the nature of which I find that the answer ought to be *less* than the second term; I therefore mark the *greater* extreme $\frac{11}{5}$ for a divisor; having inverted $\frac{11}{5}$, I put the three terms down with signs of multiplication between, and multiply them together; the product $\frac{405}{64}$ is next reduced to its proper terms, which gives the answer. In the multiplication, the *elevens* cancel each other.

2. If $\frac{4}{9}$ of a yard of lace cost $\frac{3}{5}$ of a pound, what cost $\frac{7}{16}$ of a yard?

Stating.

$$*\frac{4}{9}yd. : \frac{3}{5}L. :: \frac{7}{16}yd.$$

OPERATION.

$$\frac{3}{5} \times \frac{7}{16} \times \frac{9}{4} = \frac{189}{320}L. = 11s.\ 9d.\tfrac{3}{4}\ answer.$$

Explanation.

Here as the terms require no reducing, I first state the question, and find that the answer will be less than the second term; I therefore mark the greater term $\frac{4}{9}$ for a divisor, which being inverted, and multiplied by the two other

terms, produces the fraction $\frac{189}{320}L.$; this reduced to its proper quantity is the answer.

3. If my friend lends me 10 guineas for 10 weeks and 3 days, how long ought I to lend him 5*l.* 4*s.* 6*d.* to acquit myself of the obligation?

Reduction.

$$10\ guineas = 10\tfrac{1}{2}L. = \frac{21}{2}L. \qquad 10w.\ 3d. = 10\tfrac{3}{7}w. = \frac{73}{7}w.$$

$$5l.\ 4s.\ 6d. = 5\tfrac{9}{40}L. = \frac{209}{40}L.$$

Stating.

$$\frac{21}{2}L. : \frac{73}{7}w. :: \frac{209^*}{40}L.$$

OPERATION.

$$\frac{21}{2} \times \frac{73}{7} \times \frac{40}{209} = \frac{30660}{1463}\ week = 20\ weeks\ 6\ days\ \frac{146}{209}\ Ans.$$

Explanation.

I find that the answer will be *greater* than the second term, consequently I mark the less extreme, and proceed as before.

4. If by walking $2\frac{3}{4}$ miles an hour I can perform a journey in $8\frac{6}{7}$ days, how many days will another who can walk $3\frac{1}{5}$ miles an hour require to go the same distance, allowing 12 hours to the day?

$$Thus,\ 2\tfrac{3}{4}m. = \frac{11}{4}m. \qquad 8\tfrac{6}{7}d. = \frac{62}{7}d. \qquad 3\tfrac{1}{5}m. = \frac{16}{5}m.$$

$$Then\ \frac{11}{4}m. : \frac{62}{7}d. :: \frac{16^*}{5}m. : \frac{11}{4} \times \frac{62}{7} \times \frac{5}{16} = \frac{1705}{224}d. =$$

$$7d.\ 7h.\ \frac{19}{56}\ the\ answer.$$

5. If $\frac{3}{7}$ of a cwt. of iron cost $\frac{3}{10}$ of a pound, what is the value of $\frac{5}{6}$ of a cwt.? *Ans.* 11*s.* 8*d.*

6. If $\frac{4}{9}$ of a yard of cambric cost $\frac{3}{5}$ of a pound, what cost $\frac{5}{8}$ of a yard? *Ans.* 16*s.* 10*d.*$\frac{1}{2}$.

7. If $3\frac{1}{2}$ ounces of gold cost 17*l.* 15*s.* 6*d.* what sum will purchase $4\frac{1}{4}$ ounces? *Ans.* 21*l.* 11*s.* 8*d.*$\frac{1}{2}$.

8. If $\frac{3}{4}$ lb. of indigo cost $\frac{11}{96}L.$ what quantity will 2*l.* 3*s.* 4*d.* buy? *Ans.* 14*lb.* 2*oz.* 14*dr.*$\frac{6}{11}$.

9. Paid 2*l.* 2*s.* 8*d.* for the carriage of 1*cwt.* 2*qr.* 14*lb.*; what sum will pay for $2\frac{11}{12}$ cwt. carried the same distance? *Ans.* 3*l.* 16*s.* 6*d.*$\frac{38}{39}$.

10. If $\frac{5}{12}$ of a calf be worth $\frac{6}{7}$ of a pound, what is the whole calf worth? *Ans.* 2*l.* 1*s.* 1*d.*$\frac{5}{7}$.

11. If 7 men can mow a hay-field in $4\frac{7}{8}$ days, how long would 4 men require to mow the same? *Ans.* $8\frac{17}{32}$ *days.*

12. Paid 11 shillings for $\frac{3}{8}$ of a gallon of brandy; what must be given for $4\frac{6}{7}$ gallons at the same rate? *Ans.* 7*l.* 2*s.* 5*d.*$\frac{5}{7}$.

13. If $\frac{2}{11}$ of a block of mahogany cost $2\frac{5}{8}L.$ what sum will purchase the remaining $\frac{9}{11}$? *Ans.* 11*l.* 16*s.* 3*d.*

14. Suppose 15 feet of plank 9 inches wide sufficient to make a side-board, how many feet of plank $1\frac{1}{4}$ foot wide will be required to make another of equal dimensions? *Ans.* 7 *feet* 6 *inches.*

15. If 9 English ells of linen cost 1*l.* 17*s.* 6*d.* what cost 10 Flemish ells? *Ans.* 1*l.* 5*s.*

16. What is the value of $5\frac{3}{4}$ fother of lead, at 16*s.* 4*d.* per cwt.? *Ans.* 4*l.* 11*s.* 6*d.*$\frac{38}{40}$.

17. A corn-chandler charges 9*l.* 5*s.* 6*d.* for a load of barley; what must be given for $3\frac{2}{7}$ bushels at that price? *Ans.* 15*s.* 2*d.*$\frac{17}{20}$.

18. Suppose 120 men can build a ship in $15\frac{7}{9}$ weeks, how long will it be building if 87 men only are employed? *Ans.* 21*w.* 5*d.*$\frac{88}{261}$.

19. If 5*l.* 6*s.* 8*d.* be paid for the use of 173*l.* 2*s.* 6*d.* what sum must be paid for the use of $310\frac{2}{10}L.$ for the same time? *Ans.* 8*l.* 19*s.* 6*d.*$\frac{69}{277}$.

20. After using $\frac{2}{5}$ of a Cheshire cheese, $\frac{2}{3}$ of the remainder sold for 13*s.* 5*d.*$\frac{1}{3}$; what was the whole cheese worth? *Ans.* 1*l.* 2*s.* 5*d.*$\frac{1}{6}$.

21. If $3\frac{4}{7}$ yards of cloth $1\frac{3}{5}$ yard wide will make a suit of clothes, what quantity of cloth $\frac{5}{9}$ yard wide will be necessary to make a similar suit? *Ans.* 10*yd.* 1*qr.* $0\frac{4}{7}$*n.*

22. If $\frac{7}{9}$ of 3 guineas will purchase $\frac{1}{8}$ of a lottery-ticket, how many tickets can be had for a thousand pounds? *Ans.* 51$\frac{1}{18}$ *tickets.*

23. Bought $\frac{5}{9}$ of a privateer, $\frac{3}{7}$ of which part I presented to my brother, and afterward sold $\frac{1}{3}$ of my remaining share for 225*l.* 10*s.*; what is the privateer worth?

24. How many yards of carpet $1\frac{1}{8}$ yard wide will be required to cover a floor $8\frac{7}{9}$ yards long, and $6\frac{5}{7}$ yards wide?

211. COMPOUND PROPORTION IN VULGAR FRACTIONS[n].

EXAMPLES.

1. If $\frac{1}{10}L.$ will pay the interest of $2\frac{4}{5}L.$ for $\frac{1}{12}$ of a year, what sum will pay the interest of $10L.$ for $\frac{5}{7}$ of a year.

$$\textit{First, } 2\tfrac{4}{5}L. = \frac{14}{5}L.$$

Stating.

$$*\frac{14}{5}L. : \frac{1}{10}L. :: 10L.$$
$$*\frac{1}{12}yr. : \text{———} :: \frac{5}{7}yr.$$

OPERATION.

$$\frac{\frac{1}{10}\times 10 \times \frac{5}{7}}{\frac{14}{5}\times\frac{1}{12}} = \frac{\frac{5}{7}}{\frac{14}{60}} = \frac{300}{98} = \frac{150}{49}L. = 3l.\ 1s.\ 2d.\tfrac{46}{49}\ \textit{Ans.}^{o}$$

[n] This rule depends on the same reasons as Compound Proportion in whole numbers, and its operations are performed in the same manner, as far as relates to the general principle, differing only as a fractional differs from an integral process.

[o] It will be found more convenient to place the fractions all in one line, with the divisors inverted; thus,
$\frac{1}{10}\times\frac{10}{1}\times\frac{5}{7}\times\frac{5}{14}\times\frac{12}{1}$ (which by cancelling, Art. 203, becomes) $=$ $\frac{150}{49}L. = 3l.\ 1s.\ 2d.\frac{46}{49}$ as above.

Explanation.

Having stated the question, and marked the divisors, &c. the complex fraction which arises from the operation is reduced to a simple one, by multiplying 5 by 60, and 14 by 7; the result is next reduced to its lowest terms, and then to its proper quantity, which is the answer.

2. If 12 men earn $2\frac{1}{3}L.$ in $4\frac{1}{2}$ days, what sum will 20 men earn in $12\frac{1}{4}$ days?

Reduction.

$$First,\ 2\tfrac{1}{3}L. = \frac{7}{3}L. \qquad 4\tfrac{1}{2}\ days = \frac{9}{2}d. \qquad 12\tfrac{1}{4}\ days = \frac{49}{4}d.$$

Stating.

$$Then\ 12^{*}m.\ :\ \frac{7}{3}L.\ ::\ 20m.$$

$$\frac{^{*}9}{2}d.\ :\ \text{———}\ ::\ \frac{49}{4}d.$$

OPERATION.

$$\frac{7}{3} \times \overset{10}{\cancel{20}} \times \frac{49}{\underset{\cancel{2}}{\cancel{4}}} \times \frac{1}{12} \times \frac{\cancel{2}}{9} = \frac{3430}{324} = 10l.\ 11s.\ 8d.\tfrac{20}{27}\ Answer.$$

3. If $3\frac{1}{2}$ yards of cloth $\frac{3}{4}$ yard wide cost $\frac{4}{5}$ $L.$ what sum will $4\frac{3}{4}$ yards of $\frac{4}{5}$ yard wide, and equal in quality, cost?

$$3\tfrac{1}{2}\ yd. = \frac{7}{2}yd. \qquad 4\tfrac{3}{4}yd. = \frac{15}{4}yd.$$

$$\frac{7^{*}}{2}\ yd.\ :\ \frac{4}{5}L.\ ::\ \frac{15}{4}yd.$$

$$\frac{3^{*}}{4}\ yd.\ :\ \text{———}\ ::\ \frac{4}{5}yd.$$

$$\frac{\cancel{4}}{\cancel{5}} \times \frac{\overset{3}{\cancel{15}}}{\cancel{4}} \times \frac{4}{5} \times \frac{2}{7} \times \frac{4}{\cancel{3}} = \frac{32}{35}L. = 18s.\ 3d.\tfrac{3}{7}\ Answer.$$

4. If 9 students spend $10\frac{7}{9}L.$ in 18 days, how much will 20 students spend in 30 days? *Ans.* $39l.\ 18s.\ 4d.\frac{60}{243}$.

5. If $100L.$ gain $4\frac{3}{4}L.$ interest in a year, what sum will gain $5\frac{1}{2}L.$ interest in $\frac{5}{6}$ of a year? *Ans.* $138l.\ 18s.\ 11d.\frac{7}{19}$.

6. A trench is required to be dug 100 feet in length, and 10 men have been employed $6\frac{3}{4}$ days in digging $49\frac{4}{5}$ feet of it; how long will 12 men require to finish the remainder? *Ans.* $5d.\ 8\frac{7}{166}h.$ *at 12 hours to the day.*

7. If $13\frac{1}{2}$ feet of deal 9 inches wide cost 3*s*. 4*d*. what sum will pay for the floor of a room $15\frac{1}{2}$ feet long and $12\frac{1}{2}$ wide? *Ans.* 3*l*. 3*s*. 8*d*.$\frac{2}{9}$.

DECIMAL FRACTIONS.

212. If unity be supposed to be divided into 10, 100, 1000, &c. equal parts indefinitely, that is, into ten parts, or any multiple of ten, that which expresses any number of such parts less than that the unit is divided into, is called a Decimal Fraction.

NOTATION OF DECIMALS.

213. It is well known, that the value of any figure is increased tenfold by removing it one place to the left hand of the place it occupies; thus 1 removed one place to the left (by placing a cipher to the right of it) becomes 10, or ten times 1; remove it one place farther, and it becomes 100, or ten times 10; remove it one place farther, and it becomes 1000, or ten times 100; and so on without end. Hence likewise it follows, that if any figure be removed in a contrary direction from the left hand towards the right, its value will be decreased tenfold at every step of such removal; thus, by removing 1 one place to the right, 1000 becomes 100, or one tenth of 1000; 100 becomes 10, or one tenth of 100; 10 becomes 1, or one tenth of 10; and, since 1 is the least whole number, this is as far as we can go in whole numbers.

214. But it is plain, that 1 may be removed step by step indefinitely towards the right; if therefore we put a point to the right hand of the place of units to distinguish it, and then place the 1 to the right hand of the point, thus .1, it is evident from what has been said that it will express one tenth of an unit, or $\frac{1}{10}$; if we remove the 1 a place farther to the right, by interposing a cipher between the point and the 1, thus .01, this will express one tenth of $\frac{1}{10}$, or one hundredth part of unity; if two ciphers be interposed, thus .001, it will express one tenth of

$\frac{1}{100}$, or one thousandth part of unity; if three ciphers be interposed, it will express one tenth of $\frac{1}{1000}$, or one ten thousandth part of unity; and so on without end[p].

215. The point which is employed to mark the place of units is called *the decimal mark;* it separates between whole numbers and decimals; that part of any number which is to the left of the decimal mark being the whole number, and that to the right the decimal.

216. We have chosen the number 1 on account of its simplicity for the illustration of this doctrine; but the same equally holds true of all the other figures; thus, .2 expresses two tenths, or $\frac{2}{10}$; .03, three hundredths, or $\frac{3}{100}$; .004, four thousandths, or $\frac{4}{1000}$, &c.

217. Hence it appears, that a decimal fraction is expressed by the numerator only; the denominator, (being understood to consist of an unit with as many ciphers subjoined as the decimal has places,) is always omitted; thus, .5 denotes $\frac{5}{10}$, .07 denotes $\frac{7}{100}$, .0009 expresses $\frac{9}{10000}$, &c.; by this simple artifice, fractions and whole numbers are exhibited under one and the same form; a whole number and a decimal constituting together but one number, in all respects (the decimal mark excepted) similar to a whole number; so that if a whole number and a decimal be connected, they constitute one uniform scale, the steps of which, beginning at the right hand figure, regularly increase through both, up to the left hand figure in a tenfold proportion at each step, and decrease in the same proportion at every step through both, from the left hand figure to the right.

218. Another advantage follows from the similarity of decimals to whole numbers, namely, that the modes of operation are

---

p Hence it follows, that every cipher on the left hand in any decimal expression decreases the value of the decimal tenfold, and therefore such ciphers when they occur must always be put down, otherwise the value of the decimal will, for every cipher omitted, be increased tenfold, &c. more than its proper value.

It appears also that ciphers on the right hand of a decimal *do not* alter its value, for .5 = .50 = .500, &c. that is, $\frac{5}{10}$, $\frac{50}{100}$, $\frac{500}{1000}$, are equal to each other; this is evident, for each is equal to $\frac{1}{2}$; and the same is true in general.

alike in both; the only peculiarity in decimals relates to the right placing of the decimal mark, which will easily be understood from the rules and explanations which follow.

219. In addition to the foregoing observations, it has been thought necessary to subjoin the following table, whereby the whole system of notation in whole numbers and decimals will be satisfactorily shewn.

WHOLE NUMBERS.											DECIMALS.								
&c. &c.	Hund. of Millions.	Tens of Millions.	Millions.	Hund. of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	UNITS.	*Decimal mark.*	Tenths.	Hundredths.	Thousandths.	Tens of Thousandths.	Hund. of Thousandths.	Millionths.	Tens of Millionths.	Hund. of Millionths.	&c. &c.
&c.	9	6	8	4	7	2	5	3	1	.	3	5	2	7	4	8	6	9	&c.

It appears, by inspection of this table, that the figure on each side of the units place, and equally distant from it, is of like denomination, one in *parts*, and the other in *wholes;* but as in reading large whole numbers we make use of an abbreviated mode of expression, the same is found convenient in reading decimals. The above table is thus read; Nine hundred and sixty-eight millions, four hundred and seventy-two thousands, five hundred and thirty-one, and *thirty-five millions, two hundred and seventy-four thousands, eight hundred and sixty-nine, hundred millionths:* this, as far as it relates to the whole numbers, is sufficiently obvious. With respect to the decimals, this latter mode of expressing them, and that in the table, may at first sight appear to be different; we will shew that they are exactly of the same import.

Thus, 3 tenths .... *are equal to*	30	millions	Hundred Millionths.
5 hundredths	5		
2 thousandths	200	thousands	
7 tens of thousandths...........	70		
4 hund. of thousandths	4		
8 millionths................	800		
6 tens of millionths	60		
9 hundreds of millionths.........	9		

That is, the decimals as expressed in the table are together equal

to 35 millions, 274 thousands, 869 hundred millionths, as we expressed their value above; which was to be shewn.

220. There is another method of reading decimals, which is very convenient for practice, as follows:

The first place of decimals next to unity is called the place of *primes*; the next place is called the place of *seconds*, &c. and the figures occupying those places are called respectively *primes, seconds, thirds, fourths*, &c.

Thus the number 5.27 is read *five, two primes, and seven seconds*: and nine thirds, eight fourths, and six sevenths, is thus expressed in figures, .0098006.

221. EXAMPLES IN DECIMAL NOTATION AND NUMERATION.

Write in words the following decimals.

.8	.03	12.008
.25	.002	2.2222
.752	.1013	.00001
1.234	.2002	.12345

Write in figures the following decimals.

Six tenths. Thirty-nine hundredths. Four hundred and fifty-six thousandths. One millionth. Five, and twenty-three hundredths. Three thousand three hundred and thirty-three ten thousandths. One, and twenty-four ten thousandths. Seven, and eight primes. Two, and two seconds. Nine thirds, eight fourths, seven fifths, and six sixths. Seven sixths, eight ninths, and nine tenths.

222. ADDITION OF DECIMALS.

RULE. Place the numbers so that the decimal marks may stand in a line under each other; then will units stand under units, tens under tens, tenths under tenths, hundredths under hundredths, &c. then, beginning at the right hand, add the numbers together like whole numbers; and from the right hand of the sum cut off as many figures by the decimal mark as are equal to the greatest number of decimal places in any of the given numbers[q].

[q] The rule for placing the numbers to be added is extremely obvious; for, since figures of different denominations cannot be added together, it is plain,

EXAMPLES.

1. Add 2.34 + 35.2 + .7831 + 1.2481 + 8.0379 together.

OPERATION.

	2.34
	35.2
	.7831
	1.2481
	8.0379
Sum	47.6091
	45.2691
Proof	47.6091

Explanation.

Having placed the numbers so that like places may stand under like, I add up exactly as whole numbers are added; then I count four (equal to the greatest number of decimal places) from the right hand of the sum, and place the decimal mark to the left of the fourth figure. The proof is the same as in simple Addition.

2. Add 12.9 + 7.38 + 8.2 + .945 + 1.805 together. *Sum* 31.23.

3. Add 7.239 + 10.0046 + 3.27 + .89 + .0073 together. *Sum* 21.4109.

4. Add 942.64 + 2.301 + 71.5 + 8.457 + 3091.9 together. *Sum* 4116.798.

5. Add .4937 + .008 + .37042 + .89139 + 1.290037 together. *Sum* 3.053547.

6. Add 3748.2 + 9.8073 + 120.965 + 1374.7 + 48. together.

223. SUBTRACTION OF DECIMALS.

RULE. Place the less number below the greater, with the decimal marks under each other, so that units may stand under units, tenths under tenths, &c. as in Addition; then subtract as in whole numbers, and cut off from the right hand of the remainder as many figures for decimals as there are decimal places in either of the two given numbers [r].

---

that those of the same denomination must be placed under each other, as they alone are capable of being added.

With respect to the operation, it is plain that 10 hundredths make 1 tenth, 10 tenths make 1 unit, 10 units 1 ten, and so on in every denomination, whether it be above or below unity; wherefore, since the same law obtains in both decimal parts and whole numbers, both must evidently be added by the same rule, namely, by simple Addition.

[r] The observations contained in the preceding note apply equally to this rule, which is obvious from the nature of simple Subtraction.

When ciphers occur in one or more of the left hand *decimal* places, they

EXAMPLES.

	1.	2.	3.
From	13.745	.470349	3.617
Take	10.123	.46712	1.71438
Diff.	3.622	.003229	1.90262
Proof	13.745	.470349	3.617

4. From .740352 take .214091. *Diff.* .526261.
5. From 4.21304 take 1.20037. *Diff.* 3.01267.
6. From 12.3456 take .78095. *Diff.* 11.56465.
7. From .081059 take .0003741. *Diff.* .0806849.
8. From 987.6 take .05432. *Diff.* 987.54568.
9. From .638176 take .03749. *Diff.* 600686.

224. MULTIPLICATION OF DECIMALS.

RULE I. Place the factors so that the right hand figure of the multiplier may stand under the right hand figure of the multiplicand, and multiply as in whole numbers.

II. Count the decimal places in both factors, and from the right hand of the product mark off as many figures for decimals as there are decimals in both factors together.

To prove the operation, multiply the multiplier by the multiplicand, and proceed as before; or cast out the nines, as in simple multiplication.

III. When the number of decimals in both factors exceeds the number of figures in the product, prefix as many ciphers to the left of the product as will make up the number, and to the left of them place the decimal mark *.

must always be put down, but ciphers occupying the right hand *decimal* places may be omitted; thus in ex. 2, the two ciphers at the left of the difference are put down, and in ex. 3, the two that arise at the right hand of the proof are omitted: likewise when any of the *right hand* places of decimals are wanting, as in ex. 2 and 3, the operation is to be performed as though there were ciphers in those vacant places.

* To make the truth of this rule plain, recourse must be had to an easy example; thus, let .5 be multiplied by .3; these numbers are equivalent to $\frac{5}{10}$ and $\frac{3}{10}$, as appears from the nature of decimal notation; now $\frac{5}{10} \times \frac{3}{10}$

To prove the operation, multiply the multiplier by the multiplicand, and mark off decimals in the product as before; or cast out the nines, as in simple multiplication.

EXAMPLES.

1. Multiply 12.34567 by 3.5847.

OPERATION.

```
Thus, 12.34567
       3.5847
      -------
      8641969
     4938268
    9876536
   6172835
  3703701
 -----------
Prod. 44.255523249
```

```
  0
1   0
  0
```

Proof.

```
       3.5847 multiplier.
     12.34567 multiplicand.
     --------
       250929
      215082
     179235
    143388
   107541
   71694
  35847
  ------------
  44.255523249 Prod.
```

Explanation.

Here are 5 decimals in one factor, and 4 in the other, that is, 9 in both; I therefore count 9 places from the right of the product, and put the decimal mark to the left of the ninth place.

2. Multiply 7.38142 by .000078.

OPERATION.

```
Thus, 7.38142
      .000078
      -------
      5905136
     5166994
  -------------
Prod. .00057575076
```

Proof.

```
  6
7   6
  6
```

Explanation.

Here the factors contain 11 decimals; 3 ciphers are therefore prefixed to the product to make up 11.

$= \frac{15}{100}$ by Vulgar Fractions; also $.5 \times .3 = .15$ by the rule: but $\frac{15}{100} = .15$ by Decimal Notation; that is, the result obtained by this rule, and that obtained by Vulgar Fractions, are the same: the rule is therefore true.

If any doubt should remain respecting the truth of the rule when there are whole numbers concerned, let the factors in ex. 1. be turned into vulgar fractions and multiplied; thus $12.34567 = \frac{1234567}{100000}$, and $3.5847 = \frac{35847}{10000}$; wherefore by multiplication $\frac{1234567}{100000} \times \frac{35847}{10000} = \frac{44255523249}{1000000000} = 44\frac{255523249}{1000000000}$ $= 44.255523249$, as in ex. 1.

3. Multiply 4.82 by 3.53. *Prod.* 17.0146.
4. Multiply 47.35 by 3.74. *Prod.* 177.0890.
5. Multiply 2.1305 by .748. *Prod.* 1.5936140.
6. Multiply .056047 by 9.23. *Prod.* .51731381.
7. Multiply .2365 by .002435. *Prod.* .0005758775.
8. Multiply 9.0087 by .0000395.

225. *When the multiplier is a whole number, consisting of an unit with ciphers subjoined.*

RULE. Remove the decimal mark as many places to the right as there are ciphers in the multiplier[t].

9. Multiply 123.4567 by 10. *Prod.* 1234.567.
10. Multiply .98765 by 100. *Prod.* 98.765.
11. Multiply .00001 by 100000. *Prod.* 1.

226. To contract the operation, so as to retain in the product as many decimals only as may be thought necessary.

RULE I. Count off from the left hand of the *decimals* in the multiplicand, as many figures as are intended to be reserved in the product; and put a point over the last of these.

II. Place the units figure of the multiplier under the pointed figure, then write down the rest of the multiplier so, that the whole may stand in an inverted order, viz. the last figure first, and the first last.

III. In multiplying, always begin at that figure in the multiplicand which stands one place to the right of the multiplying figure, and carry 1 for all numbers from 5 to 15, 2 from 15 to 25, 3 from 25 to 35, &c.; but this mode of carrying is to be observed only in the first place, namely, to the figure over the multiplying figure, the product of which is the first you set down; for the rest, you are to set down and carry in the usual way.

IV. Place the several products so that all the right hand figures may stand under each other in a line; add up the pro-

---

[t] The value of any figure is increased tenfold, an hundredfold, a thousandfold, &c. by its being removed one, two, three, &c. places to the left, or (which is the same thing) by removing the decimal mark so many places to the right, as appears from Art. 18; wherefore the rule is plain.

ducts, and mark off from the right hand as many decimals as were proposed to be reserved[a].

Operations in this rule are proved by common multiplication. Art. 224.

12. Multiply 25.374856 by 5.35647, reserving only five decimal places in the product.

OPERATION.	*Proof.*	
25.374856	25.374856	
74653.5	5.35647	
12687428	177	623992
761246	1014	99424
126874	15224	9136
15224	126874	280
1015	761245	68
177	12687428	0
135.91964	135.91965	491832

Explanation.

Beginning at the decimal mark, I count 5 decimals; over the fifth I put a dot, and place the units figure 5 of the multiplier under the dotted figure, and dispose of the other figures so that the multiplier may stand *backwards*. I then begin, 5 times 6 are 30; put nothing down, but carry 3; then 5 times 5 are 25 and 3 are 28; put down 8, and, carrying the 2, proceed through the whole line as usual. For the second line, I begin 3 times 5 are 15; put nothing down, but carry 2; then I multiply the 8, carry 2, and set down throughout this line as in the first line. In the third line, I begin by multiplying the 8 for carrying, but set down the product of the 4; in the fourth, I begin at the 4, and set down at the 7; in the fifth, I begin at the 7, and set down at the 3; in the sixth line, I begin with the 3, and set down the product of the 5.

---

[a] When the factors contain a great number of decimal places, and but few are required in the product, much labour and time will be saved by the application of this rule; but care should be taken to work for one or two figures more than are wanting, as the right hand decimal arising from the contracted operation will sometimes unavoidably be wrong.

The reason of placing the units figure of the multiplier under the figure to be reserved is this, namely, that the right hand figure in every product is of the same denomination with that under which the said units figure of the multiplier stands.

The reason for reversing the multiplier will appear by consulting the operation, and comparing it with the proof; it will be seen that the first, second, third, &c. lines from the top in the former, are respectively equal to the first, second, third, &c. from the bottom in the latter.

The reason for the increase in carrying to the first figure in each line is, that the deficiency arising from the loss of what would be carried in the multiplication and addition of the figures omitted may be compensated as nearly as possible.

13. Multiply 1234.56789 by .3697428, leaving only 4 places of decimals in the product.

OPERATION.

```
         1234.56789
          8247963..
         ----------
          3703703
           740740
           111110
             8642
              494
               25
               10
         --------
Product  456.4724
```

Explanation.

The units place of the multiplier should fall under the 8, and therefore the highest figure being the first place of decimals must fall under the 7.

14. Multiply 2.38645 by 8.2175, retaining only 4 decimal places in the product. *Prod.* 19.6107.

15. Multiply 128.678 by 38.24, retaining one decimal place only in the product. *Prod.* 4920.5.

16. Multiply 325.1234567 by 23.987654, with three decimals only in the product. *Prod.* 7798.948.

17. Multiply .374853 by .0031245, with 7 decimals in the product. *Prod.* .0011713.

227. DIVISION OF DECIMALS.

RULE I. Divide as in whole numbers, then count the decimal places in the dividend, and also in the divisor, and mark off as many decimals from the right hand of the quotient as the former exceeds the latter.

II. If there are not figures enough in the quotient, add as many ciphers to the left hand as will make up the difference.

III. When there is a remainder, the quotient may be carried to any length, by bringing down ciphers, and continuing the division; but the ciphers brought down must be considered as decimals belonging to the dividend, and must be counted with those which actually stand in the dividend, in order to estimate the number of decimals to be marked off in the quotient[x].

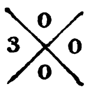
[x] The truth of this rule may be shewn by Division of Vulgar Fractions; thus, let .2464 be divided by .4; these numbers reduced to fractions are $\frac{2464}{10000}$ and $\frac{4}{10}$; therefore, inverting the divisor, and multiplying, we shall have

EXAMPLES.

1. Divide 44.80515 by 3.45.

OPERATION.

```
3.45)44.80515(12.987 quotient.
     345
     ----
     1030
      690
     ----
     3405
     3105
     ----
      3001
      2760
      ----
       2415
       2415
       ----
Proof 44.80515
```



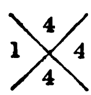
Explanation.

The division being performed in the same manner as in whole numbers, I find there are 5 decimals in the dividend and 2 in the divisor; 5 exceeds 2 by 3, I therefore mark off 3 from the right of the quotient for decimals.

2. Divide .000064 by .01234.

OPERATION.

```
.01234).00006400(.00518638 &c. quotient.
         6170
         ----
         2300
         1234
         -----
         10660
          9872
          ----
           7880
           7404
           ----
            4760
            3702
            -----
            10580
             9872
             ----
Remainder     708
```



Explanation.

Beginning at the first significant figure 6, I find that two ciphers must be added on to the dividend; I afterwards bring down ciphers, and continue the operation as far as is thought necessary. Then 5 ciphers brought down added to 8 decimals in the dividend make 13: now there are 5 decimals in the divisor, therefore 5 from 13 and 8 remains to be marked off in the quotient; but there are only 6 figures; I therefore add on 2 ciphers to the left, and prefix the decimal mark.

$\frac{2464}{10000} \times \frac{10}{4} = \frac{24640}{40000} = \frac{616}{1000} = .616$, by Art. 217; but .2464 divided by .4, according to the rule, gives also .616; wherefore this rule agreeing with one, the truth of which is established, is shewn to be right.

But it is not necessary to have recourse to Vulgar Fractions; the rule is plain from the nature of simple Division, except the right placing of the decimal mark in the quotient, which may be thus explained: if the quotient be multiplied by the divisor, with the remainder added in, the result will be the dividend; whence, by multiplication, the dividend will have as many decimal places as there are in the divisor and quotient together; wherefore the quotient must contain as many decimal places as the number of decimals in the dividend exceeds that in the divisor; which was to be shewn.

3. Divide 17.0146 by 3.53. *Quot.* 4.82.
4. Divide .51731381 by 9.23. *Quot.* .056047.
5. Divide 123.4531579 by 2.5728. *Quot.* 47.98396, &c.
6. Divide .0089 by .09876. *Quot.* .09011745, &c.
7. Divide 2508.928065051 by 92.41035.

228. *To contract the operation.*

RULE. Take as many of the left hand figures of the divisor as the quotient is intended to consist of; divide by these only, at the first step of the operation, and at each succeeding step cut off one figure (or more figures if necessary[y]) from the divisor, instead of bringing down from the dividend, using only the figures not cut off; in multiplying these by the quotient figure, you must observe to carry from the product of the preceding figure cut off, in the same manner you did in contracted multiplication, viz. 1 from 5 to 15, 2 from 15 to 25, &c. but carry as usual after you begin to set down.

To determine the place of the decimal mark in the quotient, observe under what figure of the dividend the units place of the product of the divisor by the quotient figure stands, and that will be the value of the first figure in the quotient; if it stand under units, the first figure only will be a whole number; if under tens, two figures will be whole numbers; if under primes, the first figure will be primes; if under seconds, a cipher must be prefixed to the quotient, &c.[z]

---

[y] Observing to put a cipher in the quotient whenever a figure is cut off, and the remaining figures will not go in the number to be divided.

[z] The place of the decimal mark in the quotient may be known two ways, viz. either from the decimals, or from the whole numbers; the former has been explained, and the latter may be shewn as follows. Let 10.9 divide 1234.56; now it is plain that if we had only the integers to operate with, (namely, 1234 to divide by 10,) the quotient would be 123, that is, the highest denomination 1 is of the same denomination with the figure 2 of the dividend under which the units figure 0 of the divisor stands, namely, (in this instance,) hundreds; wherefore, in the above example, seeing the units place (0) in actual division stands under the *hundreds* place (2), the highest place in the quotient will evidently be *hundreds.* Again, let 99.9 divide 1.1000; now 99 will not divide 1 (unit), and therefore will not produce *units* in the quotient; it will not divide 11 tenths, and therefore will not produce *tenths;* but it will divide 110 hundredths, and therefore will produce hundredths, or, the first or

8. **Divide 89.12543 by 12.34567, reserving only 5 figures in the quotient.**

OPERATION.	*Common method.*
12.345|67)89.12543(7.2192	12.34567)89.125|43(7.2191 &c.
.... 86 419	86419|69
2706	2705|740
2469	2469|134
237	236|6060
123	123|4567
114	113|14930
111	111|11103
3	2|038270
2	1|234567
Rem. 1	*Rem.* |803703

Explanation.

First, I cut off 2 figures from the divisor, leaving 5 on the left hand to divide by; this divisor goes 7 times in the 5 left hand figures of the dividend; I therefore begin, 7 times 6 are 42; carry 4; then 7 times 5 are 35 and 4 are 39; put down 9, carry 3, and proceed to multiply and set down in the usual way. Next I subtract, and 2706 remains; then I cut off one more figure, viz. 5, from the divisor, leaving 1234 to divide by; this goes twice in 2706; then twice 5 are 10; carry 1; twice 4 are 8 and 1 are 9; put down 9, and proceed as in common division. Next I cut off the 4, then the 3, and so on cutting off 1 figure at every step, and carrying from the figure cut off, 1 from 5 to 15, &c. as I did in contracted multiplication. The units place of the product of the divisor multiplied by 7, falling under the 9 (or units place), shews that 7 must be considered as standing in the place of units.

---

highest figure of the quotient will be hundredths, that is, of the same name with the place under which the units place of the divisor stands; and the same in other instances.

9. Divide 357.6543218 by 27.1234567; let there be 7 figures in the quotient.

```
27.12345|67)357.6543|218(13.18616 quotient.
 . . . . . .  271 2346
              864197
              813704
               50493
               27123
               23370
               21698
                1672
                1627
                  45
                  27
                  18
                  16
                   2 rem.
```

10. Divide 23.41005 by 7.9863. *Quot.* 2.9312.
11. Divide .019876843 by .012345678. *Quot.* 1.6100244.
12. Divide 721.17562 by 2.257432. *Quot.* 319.467.
13. Divide 165.6994 by 52.7438. *Quot.* 3.14159.
14. Divide 357.6543 by 13.18616. *Quot.* 27.12347.

229. In division, the products of the divisor into the several quotient figures need not be set down; each figure of any product as it arises may be subtracted from the figure under which it should stand, (if set down,) and the remainder set underneath, bringing down successively the figures of the dividend in order, or cutting off those of the divisor; observing to carry for the multiplication and subtraction both in one, whenever the carrying for both occurs [a].

---

[a] This is usually called *the Italian method*, and differs from the common method only as this is a *mental* and that a *visible* operation. Let no one suppose himself master of division until he can readily work examples in this rule, both by the common and contracted way, as is shewn in ex. 16.

15. Divide 123.456789 by .432.

OPERATION.

```
.432)123.456789(285.779604 &c. quotient.
      3705
       2496
        3367
         3438
          4149
           2610
            1800
     Rem.     72
```

Explanation.

I first find that 432 goes twice in 1234; I then say, twice 2 are 4; 4 from 4 and 0; put down 0, and carry 1; twice 3 are 6; 6 from 13 and 7; put down 7, and carry 1 for the borrowing; twice 4 are 8 and 1 are 9; 9 from 12 and 3 remain.

To this second line bring down 5; then 432 in 3705 will go 8 times; then 8 times 2 are 16; 6 from 15 and 9; carry 2, (viz. 1 for multiplying and 1 for subtracting); 8 times 3 are 24 and 2 are 26; 6 from 10, and 4; carry 3, (viz. 2 for multiplying and 1 for subtracting); 8 times 4 are 32 and 3 are 35; subtract this from 37 and 2 remain; to the remainder 249 bring down 6, and proceed as before.

16. Divide 791.0312345 by 35481.7.

Common method.

```
35481.7)791.0312345(.022294
           813972
           1043383
             3337494
              1441415
                 22147 rem.
```

Contracted method.

```
35481.7)791.0312345(.022294
           81397
           10434
            3338
             145
               3 rem.
```

17. Divide 17.0146 by 4.82. *Quot.* 3.53.

18. Divide 4.5172834 by 12.34. *Quot.* .366068 &c.

19. Divide .0064 by .51863. *Quot.* .01234 &c.

20. Divide 2508.928065051 by 27.1498. *Quot.* 92.41035.

230. *When the divisor is a whole number, consisting of an unit, with ciphers subjoined.*

RULE. Remove the decimal mark as many places to the left hand as there are ciphers in the divisor [b].

21. Divide 123.45 by 10. *Quot.* 12.345.

22. Divide 9876.54 by 10000. *Quot.* .987654.

23. Divide 1 by one million. *Quot.* .000001.

---

[b] The truth of this appears from decimal notation; it may likewise be proved by actually dividing, and marking off for decimals in the quotient, according to the common method, Art. 227.

REDUCTION OF DECIMALS.

231. Reduction of decimals teaches to change decimal fractions from one form to another, without altering their value.

232. *To reduce a decimal to a vulgar fraction.*

RULE I. Under the given decimal write 1, with as many ciphers subjoined as the decimal has places for a denominator.

II. Reduce this fraction to its lowest terms for the answer[c].

EXAMPLES.

1. Reduce .24 to a vulgar fraction.

Thus, 100 *the denominator.*

Then $\frac{24}{100} = \frac{6}{25}$ *the vulgar fraction required.*

2. Reduce .5 to a vulgar fraction. *Ans.* $\frac{1}{2}$.

3. Reduce .625 to a vulgar fraction. *Ans.* $\frac{5}{8}$.

4. Reduce .02525 to a vulgar fraction. *Ans.* $\frac{101}{4000}$.

233. *To reduce a vulgar fraction to a decimal.*

RULE I. Subjoin as many ciphers as are necessary to the numerator, and place the decimal mark between the numerator and those ciphers.

II. Divide this number by the denominator, and the quotient, with the decimal mark prefixed, will be the answer.

5. Reduce $\frac{7}{8}$ to a decimal.

OPERATION.

8)7.000

.875 *the answer.*

Explanation.

I add 3 ciphers as decimals to the numerator 7, and divide by the denominator 8, marking off decimals by the rule, Art. 227.

---

[c] This rule will be easily understood, as it is a natural consequence of the decimal mode of notation.

6. Reduce $\frac{8}{9}$ to a decimal.

Thus, 9)8.000
.888 &c. *Ans.*

7. Reduce $\frac{1}{13}$ to a decimal.

Thus, 13)1.00(.076923 &c. *Answer.*
90
120
30
40
1

8. Reduce $\frac{1}{2}$ to a decimal. *Ans.* .5.

9. Reduce $\frac{1}{4}$ to a decimal. *Ans.* .25.

10. Reduce $\frac{3}{4}$ to a decimal. *Ans.* .75.

11. Reduce $\frac{7}{12}$ to a decimal. *Ans.* .5833 &c.

12. Reduce $\frac{101}{102}$ to a decimal. *Ans.* .990196 &c.

234. *To reduce money, weights, and measures, to decimals.*

RULE I. Reduce the given number to a fraction of the integer of which it is to be made a decimal, by Art. 185.

II. Reduce this fraction to a decimal by the last rule[d].

13. Reduce 2*s.* 6*d.* to the decimal of a pound.

Thus, 2*s.* 6*d.* $= \frac{1}{8}$*L.* *Then* 8)1.000
.125*L. the answer.*

14. Reduce 1*qr.* 14*lb.* to the decimal of a cwt.

Thus, 1*qr.* 14*lb.* $= \frac{3}{8} =$ .375*cwt. Answer.*

15. Reduce 5*bu.* 3*pks.* to the decimal of a chaldron.

Thus, 5*bu.* 2*pks.* $= \frac{11}{72} =$ .152777 &c. *chald. Ans.*

---

[d] The foundation of this rule is sufficiently obvious without explanation.

16. Reduce 5*s.* to the decimal of a pound. *Ans.* .25*l.*

17. Reduce 3*s.* 6*d.* to the decimal of a guinea. *Ans.* .1666 &c. *guinea.*

18. Reduce 3*oz.* 5*dwts.* to the decimal of a lb. troy. *Ans.* .27083 &c. *lb.*

19. Reduce 2*r.* 20*p.* to the decimal of an acre. *Ans.* .625 *acre.*

235. *When the given quantity is of several denominations.*

RULE I. Write all the given denominations in a line under each other, beginning with the least, and proceeding in order up to the greatest.

II. On the left of each, place that number for a divisor which will reduce it to a decimal of the next superior denomination.

III. Divide each of the denominations, together with the decimals which arise, by its proper divisor, and the last quotient will be the decimal required [e].

20. Reduce 4*s.* 6*d.* $\frac{3}{4}$ to the decimal of a pound.

OPERATION.

4	3.00
12	6.75
20	4.5625
	.228125*l.* *Ans.*

Explanation.

I put down 3 farthings first, 6*d.* next, and 4*s.* last; opposite these on the left I place 4, 12, and 20, being respectively the divisors which will reduce each to decimals of a superior denomination; I then divide each line, ciphers being subjoined where they are required.

21. Reduce 5℥ 4ʒ 2℈ 1*gr.* to the decimal of a lb.

20	1.00
3	2.05
8	4.68333333
12	5.58541666
	.46545138 *lb. Answer.*

[e] By this process the least denomination is reduced to a decimal of the next superior denomination, as is obvious from the foregoing rule; this latter denomination, with the said decimal subjoined, is reduced to a decimal of the next superior denomination; this again with its decimal to a decimal of the next; and so on to the highest: for instance, in ex. 20, $\frac{3}{4}$ is reduced to the decimal of a penny, by dividing the 3 by the 4; then 6*d.*, with this decimal, (viz. 6.75,) is reduced to the decimal of a shilling, by dividing it by 12; lastly, the whole number, with this decimal, (viz. 4.5625,) is reduced to the decimal of a pound; which reductions are true, according to Art. 233; and the same of other examples: wherefore the rule is manifest.

22. Reduce 3*s*. 2*d*.¼ to the decimal of a pound. *Answer* .159375*l*.

23. Reduce 7*d*.½ to the decimal of a shilling. *Ans*. .625*s*.

24. Reduce 1*s*. 2*d*.¾ to the decimal of a crown. *Answer* .245833*cr*.˙

25. Reduce 2*qr*. 3*n*. to the decimal of a yard. *Ans*. .6875*yd*.

26. Reduce 1*r*. 14*p*. to the decimal of an acre. *Ans*. .3375*a*.

27. Reduce 4*bu*. 2*pks*. 1*gal*. to the decimal of a quarter. *Ans*. .578125*qr*.

28. Reduce 1*qr*. 2*lb*. 3*oz*. 4*dr*. to the decimal of a cwt. *Ans*. .2696707 &c. *cwt*.

29. Reduce 3*w*. 4*d*. 5*h*. 6*m*. 7″ to the decimal of a month. *Ans*. .9004493 &c. *month*.

236. *To reduce a decimal to its proper quantity.*

RULE I. Multiply the given decimal by that number which will reduce it to the next lower denomination, and mark off as many decimals from the product as there are decimal places in the given number.

II. Multiply these decimals by the number which will reduce them to the next lower denomination, and mark off decimals as before.

III. Proceed in this manner until you have reduced the decimals to the lowest denomination possible; then all the whole numbers being collected, and placed in order, will be the answer[f].

30. Reduce .228125 of a pound to its proper quantity.

OPERATION.

$$\begin{array}{r}
.228125\ L. \\
20 \\ \hline
4.562500\ s. \\
12 \\ \hline
6.750000\ d. \\
4 \\ \hline
3.000000\ qrs.
\end{array}$$

Explanation.

I multiply the given decimal (of a pound) by 20, and mark off 6 places; these I multiply by 12, and mark off 6 places; these I multiply by 4, and again mark off 6 places. Then of the whole numbers, the 4 will evidently be shillings, the 6 pence, and the 3 farthings, which together are the answer.

Ans. 4*s*. 6*d*.¾.

[f] This rule has its origin in the nature of Compound Division; for if the given decimal be considered as a remainder, the successive multiplications, and marking off decimals, are equivalent to the successive reducing and dividing in that rule.

31. Reduce .7583 of a lb. troy to its proper quantity.

```
 .7583 lb.
    12
 ------
9.0996 oz.
    20
 ------
 1.9920 dwts.
    24
 ------
 39680
19840                Ans. 9oz. 1dwt. 23gr.
 ------
23.8080 grains.
```

32. Reduce .159375 of a pound to its proper quantity. *Ans.* 3*s.* 2*d.*$\frac{1}{4}$.

33. Reduce .625 of a shilling to its proper quantity. *Ans.* 7*d.*$\frac{1}{2}$.

34. Reduce .05854 of a guinea to its proper quantity. *Ans.* 1*s.* 2*d.*$\frac{3}{4}$.

35. Reduce .6875 of a yard to its proper quantity. *Ans.* 2*qr.* 3*n.*

36. What is the value of .625 cwt.? *Ans.* 2*qr.* 14*lb.*

37. Reduce .3375 of an acre to its proper quantity. *Ans.* 1*r.* 14*p.*

38. What is the value of .461 of a chaldron? *Answer* 16*bu.* 2*pks.*

39. What is the value of .85714 of a month? *Ans.* 3*w.* 2*d.* 23*h.* 59*m.* 53″.

237. Promiscuous Examples for Practice.

1. What is the value of $.135 + .243 \times .312$? *Ans.* .210816.

2. What is the value of $.0098 - .00098 \times .54$? *Answer* .0092708.

3. Required the value of $3.74 \times 2.35 - 1.23$. *Ans.* 7.559.

4. Required the value of $1.35 \times \overline{3.8 + .019}$. *Ans.* 5.15565.

5. What is $2.04 \times \overline{4.5 - 2.95}$ equal to? *Ans.* 3.162.

6. What is $\overline{1.23 + .94} \times 2.12 - .89$ equal to? *Ans.* 3.7104.

7. What is $\overline{1.23 + .94} \times \overline{2.12 - .89}$ equal to? *Ans.* 2.6691.

8. Find the value of $\overline{3.4 - 2.5} \times \overline{.12 + 5.1 - .12}$. *Ans.* 4.578.

9. Find $47.5 - 3.7 \times \overline{.23 - .017} - 1.0734$. *Ans.* 45.6385.

10. Find $\dfrac{12.3 \times 2.34 \times 3.12}{123.4}$. *Ans.* .7277134 &c.

11. Required $\dfrac{48.5 + 3.14 - 21.35}{43.21}$. *Ans.* .7009951 &c.

12. To determine $2.473 + \dfrac{3.82 \times .012}{36.5}$. *Ans.* 2.474255 &c.

13. Find $\dfrac{81.2 - 34.7 \times .043}{10.8}$. *Ans.* 7.380361 &c.

14. Required $\dfrac{1.5 \times 4.2 - 1.01 \times 2.13}{3.5 \times .12 - .1234}$ being the decimal of a pound, to be reduced to its proper quantity. *Ans.* 13*l.* 19*s.* 9*d.* .006.

238. PROPORTION, OR, THE RULE OF THREE IN DECIMALS.

RULE. Reduce the first and third terms to decimals of the same denomination, and the second to a decimal of the greatest denomination mentioned; then state the question, and proceed as in the Rule of Three in Vulgar Fractions: the result will be of the same denomination with the second term, and (if a decimal) must be reduced to its proper quantity[g].

EXAMPLES.

1. If $\frac{3}{8}$ of a cwt. of pimento cost 3*l.* 2*s.* 6*d.* what is the value of 1*cwt.* 2*qr.* 14*lb.*?

Reduction of the terms.

$\frac{3}{8}$*cwt.*=.375*cwt.* 3*l.* 2*s.* 6*d.*=3.125*l.* 1*cwt.* 2*qr.* 14*lb.*=1.625*cwt.*

Stating. OPERATION.

$$.375cwt. : 3.125l. :: 1.625cwt. : \frac{3.125 \times 1.625}{.375} = \frac{5.078125}{.375}$$

$$= 13.54166 = 13l.\ 10s.\ 9d.\tfrac{1}{4}\ .99 \text{ \&c. } \textit{Answer.}$$

Explanation.

Having reduced the first and third terms to decimals of a cwt. and the second to the decimal of a pound, I state the question, and, finding that the answer

[g] The rules of Proportion in whole numbers, vulgar and decimal fractions, depend all on the same principles; they differ only in the different modes of operation peculiar to each of these three kinds of numbers.

ought to be greater than the second term, I mark the less extreme for a divisor; I multiply the two unmarked terms together, and divide the product by the marked term, the quotient 13.54166 &c. reduced to its proper terms is the answer.

2. If 2*yds.* 3*qrs.* of cloth $\frac{7}{8}$ yard wide will make a coat, how much shalloon half yard wide will be required to line it throughout?

Thus, 2*yds.* 3*qrs.* = 2.75*yd.* $\frac{7}{8}$*yd.* = .875*yd.* $\frac{1}{2}$*yd.* = .5*yd.*

$$.875y. : 2.75y. :: .5^*y. : \frac{.875 \times 2.75}{.5} = \frac{2.40625}{.5} = 4.8125yd.$$

= 4*yds.* 3*qrs.* 1*n.* *Answer.*

3. Sold a barrel of ale for 3*l.* 7*s.* 6*d.*; what sum will pay for 35.264 barrels? *Ans.* 119*l.* 0*s.* 3*d.*$\frac{3}{4}$ .36.

4. If 50*l.* will pay for 16*cwt.* 1*qr.* 16*lb.* of raisins, what is the price per cwt.? *Ans.* 3*l.* 1*s.* .00218.

5. Bought $\frac{3}{8}$ of a levant trader for 357*l.* 5*s.* what sum will purchase $\frac{4}{5}$ of the remainder? *Ans.* 476*l.* 6*s.* 8*d.*

6. Paid 9.87*l.* for 6.54 cwt. of stock-fish; what quantity can be had for 32.1*l.* at that rate? *Ans.* 21*cwt.* 1*qr.* 2*lb.* .2297856.

7. If a lump of ore, weighing 15.253 *lb.*, be valued at 3*s.* 9*d.* what is the cargo of a ship, carrying 180 tons of the same, worth? *Ans.* 4956*l.* 8*s.* 0*d.*$\frac{1}{4}$ .939.

8. A piece of cloth was cut into two parts, one of which measured 5$\frac{1}{4}$ English ells, and the other 8$\frac{3}{4}$ Flemish ells; what is the value of the whole, at 8*s.* 4*d.*$\frac{1}{4}$ per yard?

239. COMPOUND PROPORTION IN DECIMALS.

RULE. Prepare the numbers (if they require it) as in the preceding rule, and work as in Compound Proportion in whole numbers[h].

EXAMPLES.

1. If 7 men earn 9*l.* 10*s.* 6*d.* in $10\frac{1}{2}$ days, what sum will 28 men earn in $31\frac{1}{2}$ days.

Reduction of the terms.

9*l.* 10*s.* 6*d.*=9.525*l.* $10\frac{1}{2}$*d.*=10.5*d.* $31\frac{1}{2}$*d.*=31.5*d.*

Stating.

$$\begin{array}{l} {}^*7m. : 9.525l. :: 28m. \\ {}^*10.5d. : \text{———} :: 31.5d. \end{array}$$

OPERATION.

$$\frac{9.525 \times \overset{4}{\cancel{28}} \times \overset{3}{\cancel{31.5}}}{\cancel{7} \times \cancel{10.5}} = L.\,114.3 = 114l.\ 6s.\ \textit{Ans.}$$

2. If 30*l.* in 20 months gain 10*l.* 5*s.* what sum will 20*l.* gain in 10 months?

$$\begin{array}{l} \textit{Thus, } {}^*30l. : 10.25l. :: 20l. \\ \quad {}^*20m. : \text{———} :: 10m. \end{array}$$

$$\textit{Then } \frac{10.25 \times 20 \times 10}{30 \times 20} = 3l.\ 8s.\ 4d.\ \textit{Ans.}$$

3. If 3*yd.* 2*qr.* 1*n.* of cloth that is $\frac{1}{2}$ yard wide cost 9*s.* 6*d.*, what cost 4*yd.* 3*qr.* 2*n.* of $\frac{3}{4}$ yard wide, and of equal goodness? *Ans.* 19*s.* 6*d.*

4. Paid 3*l.* 7*s.* 4*d.* for the carriage of 5*cwt.* 3*qrs.* 150 miles; what sum will pay for the carriage of 7*cwt.* 2*qr.* 25*lb.* 64 miles at the same rate? *Ans.* 1*l.* 18*s.* 7*d.*.0525.

---

[h] This rule depends on the same principles with Compound Proportion in whole numbers.

CIRCULATING DECIMALS.

240. Circulating, repeating, or recurring decimals are those in which one or more of the figures continually recur, and may be carried on indefinitely; the figures that recur are called *repetends.*

241. A pure repetend is a decimal in which all the figures recur; as .222 &c. .012012 &c. .153153 &c.

242. A mixed repetend is a decimal in which some of the figures do, and some do not, recur; as .5333 &c. .341212 &c. .419375375 &c.

243. A single repetend is that in which only one figure repeats, as .333 &c. and is denoted by a point placed over the circulating figure, as $.\dot{3}$.

244. A compound repetend is that in which the same figures repeat alternately, as .1212 &c. .345345 &c. and is expressed by a point over the first and last repeating figure, as $\dot{1}\dot{2}$....$\dot{3}4\dot{5}$ &c.

245. Similar repetends are those which begin at equal distances from the decimal mark; *thus* $.23\dot{5}\dot{7}$, $.47\dot{1}$, *and* $.49\dot{3}85\dot{7}$, *are similar.*

246. Dissimilar repetends are those which do not begin at equal distances from the decimal mark; *thus* $.23\dot{1}2\dot{3}$ *and* $.4\dot{5}3\dot{1}$ *are dissimilar.*

247. Conterminous repetends are such as end at equal distances from the decimal mark; *thus* $.\dot{2}3232\dot{3}$ *and* $.\dot{3}1531\dot{5}$ *are conterminous, as are* $.3\dot{4}51\dot{7}$ *and* $.\dot{8}241\dot{3}$.

248. Similar and conterminous repetends are such as begin at the same distance from the decimal mark, and end at the same distance; *thus* $.785\dot{3}4343\dot{4}$ *and* $.000\dot{7}8978\dot{9}$ *are similar and conterminous, as are* $.1\dot{2}34\dot{5}$ *and* $.5\dot{4}32\dot{1}$.

REDUCTION OF CIRCULATING DECIMALS.

249. *To reduce a pure repetend to its equivalent vulgar fraction.*

RULE I. Under the given repetend as a numerator write as many nines as the repetend has figures for a denominator.

II. Reduce this fraction to its lowest terms, which will be the fraction required[i].

EXAMPLES.

1. Required the values of $.\dot{3}$ and $.\dot{3}\dot{6}$ in vulgar fractions?

Thus $.\dot{3} = \frac{3}{9} = \frac{1}{3}$ *Ans.* *and* $.\dot{3}\dot{6} = \frac{36}{99} = \frac{4}{11}$ *Ans.*

2. Reduce $.\dot{2}3\dot{4}$ and $.\dot{3}4123\dot{1}$ to equal vulgar fractions.

Thus $.\dot{2}3\dot{4} = \frac{234}{999} = \frac{26}{111}$ *Ans.* *and* $.\dot{3}4123\dot{1} = \frac{341231}{999999} = \frac{31021}{90909}$ *Ans.*

3. Reduce $.\dot{6}$ and $.\dot{4}\dot{5}$ to vulgar fractions. *Ans.* $\frac{2}{3}$ *and* $\frac{5}{11}$.

4. Reduce $.\dot{2}1\dot{3}$ and $.\dot{7}28\dot{6}$ to equal vulgar fractions. *Answer* $\frac{71}{333}$ *and* $\frac{7286}{9999}$.

5. Reduce $.\dot{1}3\dot{5}$ and $.\dot{7}6923\dot{0}$ to fractions. *Ans.* $\frac{5}{37}$ *and* $\frac{370}{481}$.

---

[i] If unity, with ciphers subjoined, be divided by 9, *in infinitum*, the quotient will be 1 continually; thus $\frac{1}{9} = .\dot{1}$; whence also $\frac{2}{9} = .\dot{2}$, $\frac{3}{9} = .\dot{3}$, $\frac{4}{9} = .\dot{4}$, &c. to $\frac{9}{9} = 1$; wherefore every single repetend is equal to a vulgar fraction, the numerator of which is the repeating figure, and the denominator 9.

In the same manner $\frac{1}{99} = .\dot{0}\dot{1}$; whence $\frac{2}{99} = .\dot{0}\dot{2}$, $\frac{3}{99} = .\dot{0}\dot{3}$, &c. also $\frac{1}{999} = .\dot{0}0\dot{1}$; whence $\frac{2}{999} = .\dot{0}0\dot{2}$, $\frac{3}{999} = .\dot{0}0\dot{3}$; and the same holds true universally. Wherefore every pure repetend is equal to a vulgar fraction, the numerator of which consists of the repeating figures, and its denominator of as many nines as there are repeating figures; which was to be shewn.

250. *When any part of the repetend is a whole number.*

RULE. Subjoin as many ciphers to the numerator as the highest place of the repetend is distant from the decimal mark[k].

6. Reduce $\dot{1}.0\dot{1}$ and $\dot{1}2.7\dot{8}$ to fractions.

Thus, $\dot{1}.0\dot{1} = \frac{1010}{999}$ *Ans.* *and* $\dot{1}2.7\dot{8} = \frac{127800}{9999} = \frac{14200}{1111}$ *Ans.*

7. Reduce $\dot{2}.4\dot{6}$, $\dot{1}8.2\dot{4}$, and $\dot{1}234.\dot{8}$ to vulgar fractions. *Answer* $\frac{820}{333}$, $\frac{60800}{3333}$ *and* $\frac{41160000}{33333}$.

8. Reduce $\dot{1}.\dot{3}$, $\dot{2}1.\dot{7}$, and $\dot{3}12.\dot{4}$ to equal vulgar fractions. *Ans.* $\frac{130}{99}$, $\frac{21700}{999}$ *and* $\frac{284000}{909}$.

251. *To reduce a mixed repetend to its equivalent vulgar fraction.*

RULE I. Prefix as many nines as there are places in the repetend, to as many ciphers as there are places in the finite part, for a denominator.

II. From the given mixed repetend subtract the finite part for a numerator, and reduce the fraction to its lowest terms for the answer.

---

[k] This rule may be explained by example 6; where if we suppose $\dot{1}0\dot{1}$ to be wholly a decimal, its equivalent vulgar fraction will be $\frac{101}{999}$, by the preceding rule; but $\dot{1}.0\dot{1}$ is ten times $\dot{1}0\dot{1}$, whence the foregoing fraction multiplied by 10, (thus $\frac{101}{999} \times 10$,) or $\frac{1010}{999}$, will be the value of $\dot{1}.0\dot{1}$. Again, if $\dot{1}27\dot{8}$ be considered as a decimal, its equivalent vulgar fraction will be $\frac{1278}{9999}$; but $\dot{1}2.7\dot{8}$ is 100 times $.\dot{1}27\dot{8}$; wherefore the vulgar fraction, equal to the former, will be 100 times as great as that equal to the latter, that is, $\dot{1}2.7\dot{8} = \frac{127800}{9999}$; which is the rule.

9. Reduce $.12\dot{3}$ to an equal vulgar fraction.

OPERATION.

First 900 *denominator.*

Then $123 - 12 = 111$ *numerator.*

Therefore $\frac{111}{900} = \frac{37}{300}$ *the fraction required.*

Explanation.

We have here one repeating figure, and two in the finite part; therefore to one 9 I subjoin two ciphers, making 900 for the denominator. Then from 123 the given mixed repetend, I subtract 12 the finite part, which gives 111 for the numerator; the fraction is then reduced to its lowest terms.

10. Reduce $.5\dot{9}2\dot{5}$ to an equivalent vulgar fraction.

Thus 9990 *denominator. and* $5925 - 5 = 5920$ *numerator.*

Therefore $\frac{5920}{9990} = \frac{16}{27}$ *the answer.*

11. Reduce $.2\dot{7}$ and $.5\dot{3}$ to vulgar fractions. *Ans.* $\frac{5}{18}$ *and* $\frac{8}{15}$.

12. Reduce $.3\dot{4}\dot{5}$, $.\dot{1}23\dot{4}$, and $.43\dot{2}1\dot{0}$ to vulgar fractions.

252. *To make dissimilar repetends similar and conterminous.*

RULE I. Consider which of the given repetends begins the farthest from unity, and continue each of the other repetends to as many places from unity, putting a dot over the right hand figure of each; this will make them similar.

II. Continue all the repetends to as many more figures as are equal to the least common multiple of the several numbers of places in all the repetends, and place a dot over the last, or right hand figure; this will make the repetends conterminous.

13. Given the following dissimilar repetends $.\dot{3}2\dot{1}$, $5.4\dot{7}$, $3.\dot{2}$, $.12\dot{3}$, and $2.\dot{3}\dot{9}$, to make them similar and conterminous.

OPERATION.

dissimilar.		*sim. and conterm.*
$.\dot{3}2\dot{1}$	=	$.32\dot{1}3213\dot{2}$
$5.4\dot{7}$	=	$5.47\dot{4}7474\dot{7}$
$3.\dot{2}$	=	$3.22\dot{2}2222\dot{2}$
$.12\dot{3}$	=	$.12\dot{3}1231\dot{2}$
$2.\dot{3}\dot{9}$	=	$2.39\dot{3}9393\dot{9}$

Explanation.

Here $.12\dot{3}$ is the repetend which begins farthest from unity; I therefore continue all the other repetends to the third place, over which I put a dot. Now one of the repetends contains 3 places, one contains 2, and two contain each 1, and the least common multiple of 3, 2, and 1, is 6; wherefore I continue the repeating figures in each repetend to 6 places farther, and place a dot over the last.

14. Given $123.4\dot{5}$, $3.9\dot{1}\dot{2}$, $.\dot{3}01\dot{2}$, and $9.\dot{3}8\dot{5}$, to make them similar and conterminous.

dissimilar.		*sim. and conterm.*
$123.4\dot{5}$	=	$123.4\dot{5}5555555555\dot{5}$
$3.9\dot{1}\dot{2}$	=	$3.9\dot{1}2121212121\dot{2}$
$.\dot{3}01\dot{2}$	=	$.3\dot{0}1230123012\dot{3}$
$9.\dot{3}8\dot{5}$	=	$9.3\dot{8}5385385385\dot{3}$

15. Make $12.\dot{3}8\dot{4}$ and $2.3\dot{4}$ similar and conterminous. *Ans.* $12.\dot{3}84\dot{3}$ *and* $2.\dot{3}44\dot{4}$.

16. Make $.123\dot{4}$ and $5.0\dot{4}3219\dot{9}$ similar and conterminous. *Ans.* $123\dot{4}444\dot{4}$ and $5.043\dot{2}194\dot{3}$.

17. Make $3.\dot{5}$, $\dot{4}\dot{5}$, and $1.\dot{0}8\dot{4}$, similar and conterminous.

253. *To find whether the decimal equivalent to any given vulgar fraction be finite or infinite; and if infinite, to find how many places the repetend will consist of.*

RULE I. Reduce the given fraction to its lowest terms, then divide the denominator of the new fraction by 10, 5, or 2, as long as division by either can be made.

II. If by this division the denominator be reduced to unity, the decimal will be finite, consisting of as many places as you performed divisions.

III. But if after such division the denominator, viz. the last quotient, be greater than unity, divide 9999, &c. by the said last quotient till nothing remains; the number of nines made use of will be equal to the number of figures in the repetend, which will begin after as many places of figures as there were divisions by 10, 5, or 2.

18. Is the decimal equivalent to $\frac{210}{1120}$ finite or infinite? if infinite, how many places does the repetend consist of?

First, $\frac{210}{1120}$ *reduced to its lowest terms is* $\frac{3}{16}$.

Then, $16 \overset{(2)}{\ldots} 8 \overset{(2)}{\ldots} 4 \overset{(2)}{\ldots} 2 \overset{(2)}{\ldots} 1$; *therefore the given fraction produces a finite decimal, consisting of four places, viz.* $\frac{210}{1120} = \frac{3}{16} = .1875$.

19. Is the decimal equivalent to $\frac{1111}{7700}$ finite or infinite? if in-

finite, where does the repetend begin, and how many places does it consist of?

First, $\frac{1111}{7700}$ *reduced to its lowest terms is* $\frac{101}{700}$.

Then, $\overset{(10)}{700} \ldots \overset{(5)}{70} \ldots \overset{(2)}{14} \ldots 7$; *wherefore* $7\overline{)999999}$ *in which* $\overline{142857}$,

six nines are used before the work terminates. Now since 3 *divisions (by* 10, 5, *and* 2*) have taken place, there will be* 3 *finite places; and since there are six nines employed in the division by the last quotient* 7, *there will be six circulating figures, beginning at the fourth place of decimals. Thus,* $\frac{1111}{7700} = \frac{101}{700} = .144\dot{2}8571\dot{4}$.

20. Are the decimals equivalent to $\frac{23}{60}$, $\frac{11}{12}$, $\frac{113}{114}$, and $\frac{975}{1050}$, finite or infinite, how many places does each consist of, and what are the particulars?

254. ADDITION OF CIRCULATING DECIMALS.

RULE I. Make the repetends which are to be added together similar and conterminous, (*Art.* 252.)

II. On the right hand of each repetend place two or three of the repeating figures, and add them together for the purpose of carrying.

III. Carry the tens from the left of the sum of these figures to the right hand row of figures in the repetends, and add up the whole as in finite decimals; then mark as many figures of the sum for a repetend as there are in each repetend added[1].

---

[1] The reason of this rule is sufficiently plain; for it is evident, that all the repetends to be added must be made similar and conterminous (if they are not so already) before the operation commences: and since these repetends may be continued indefinitely, and that the sum of the right hand figures of the first repetend would, in that case, be increased by the number carried from the left hand figures of the second, and the sum of the right hand figures of the second by the number carried from the right hand figures of the third, and so on; and that these carryings would be always the same, as each arises from the addition of the same figures; it follows, that, in order to have the true repetend in the sum, the right hand figure of that repetend must be increased by the number

EXAMPLES.

1. Add $2.\dot{7} + 12.3\dot{4}5\dot{6} + .\dot{4}\dot{5} + 456.\dot{7} + 987. + .1\dot{2}3\dot{4}$ together.

OPERATION.

dissimilar.		*similar and conterminous.*	*fig.*
$2.\dot{7}$	=	$2.7\dot{7}77777\dot{7}$	777
$12.3\dot{4}5\dot{6}$	=	$12.3\dot{4}56456\dot{6}$	456
$.\dot{4}\dot{5}$	=	$.4\dot{5}4545\dot{4}$	545
$456.\dot{7}$	=	$456.7\dot{7}77777\dot{7}$	777
$987.$	=	$987.0\dot{0}00000\dot{0}$	000
$.1\dot{2}3\dot{4}$	=	$.1\dot{2}34234\dot{4}$	234
	Sum	$1459.4\dot{7}91700\dot{0}$ *carry* 2 *to the* $\dot{4}$.	

Explanation.

The repetends being made similar and conterminous, the numbers marked *fig.* on the right are a few of the first figures of each repetend, and are added, only to find what is to be carried to the $\dot{4}$.

2. Add $78.3\dot{4}7\dot{6} + 18.\dot{6} + 735.\dot{2} + 375.1 + 187.\dot{4} + 3.\dot{2}\dot{7}$ together.

dissimilar.		*similar and conterminous.*	*fig.*
$78.3\dot{4}7\dot{6}$	=	$78.3\dot{4}76476\dot{6}$	476
$18.\dot{6}$	=	$18.6\dot{6}66666\dot{6}$	666
$735.\dot{2}$	=	$735.2\dot{2}22222\dot{2}$	222
375.1	=	$375.1\dot{0}00000\dot{0}$	000
$187.\dot{4}$	=	$187.4\dot{4}44444\dot{4}$	444
$3.\dot{2}\dot{7}$	=	$3.2\dot{7}27272\dot{2}$	727
	Sum	$1398.0\dot{5}37082\dot{2}$ *carry* 2 *to the* $\dot{2}$.	

3. Add $1.\dot{2} + .\dot{3}\dot{5} + 12.\dot{3} + 123.\dot{4} + 4.3\dot{2}$ together. *Sum* $141.\dot{6}\dot{7}5$.

4. Add $17.6\dot{4} + 2.\dot{8} + 4.2\dot{3} + 1.\dot{8}\dot{3} + 54.\dot{9}$ together. *Sum* $81.\dot{6}\dot{0}4$.

---

carried from the left; or, (which is the same,) by carrying from the numbers marked *fig.* (as in the 1st and 2d examples) to the said right hand figure of the repetend.

5. Add $87.1\dot{2} + 9.\dot{7}2\dot{8} + 2.3 + 4.\dot{5} + 8.\dot{0}\dot{8}$ together. *Sum* $111.7\dot{8}7314\dot{5}$.

6. Add $.\dot{1}2\dot{3} + 4.0\dot{5} + 71.\dot{6} + 12.9\dot{1} + 3.\dot{1}2345\dot{6}$ together. *Sum* $91.\dot{8}79912\dot{5}$.

255. SUBTRACTION OF CIRCULATING DECIMALS.

RULE I. Make the given repetends similar and conterminous, and place the less number under the greater.

II. Subtract as in finite decimals; observing, that if the lower repetend be greater than the upper, the right hand figure of the remainder must be made less by 1 than it would be were the expressions finite [m].

EXAMPLES.

1. From $123.4\dot{5}$ take $21.\dot{5}3\dot{2}$.

OPERATION.

$$\begin{array}{rcl} \textit{dissimilar.} & & \textit{similar and conterminous.} \\ 123.4\dot{5} & = & 123.4\dot{5}5555\dot{5} \\ 21.\dot{5}3\dot{2} & = & 21.5\dot{3}2532\dot{5} \\ \hline \textit{Difference} & & 101.9\dot{2}3023\dot{0} \end{array}$$

2. From $374.\dot{1}2\dot{3}$ take $40.37\dot{9}$.

OPERATION.

$$\begin{array}{rcl} \textit{dissimilar.} & & \textit{similar and conterminous.} \\ 374.\dot{1}2\dot{3} & = & 374.123\dot{1}2\dot{3} \\ 40.37\dot{9} & = & 40.379\dot{9}9\dot{9} \\ \hline \textit{Diff.} & & 333.743\dot{1}2\dot{3} \end{array}$$

Explanation.

Here the repetend to be subtracted is the greater, the right hand figure of the difference is therefore decreased by 1.

3. From $39.2\dot{1}7\dot{8}$ take $17.\dot{6}\dot{8}$. *Diff.* $21.5\dot{3}0949\dot{1}$.
4. From 1.3 take $1.004\dot{7}$. *Diff.* $.295\dot{2}$.
5. From $85.\dot{6}\dot{2}$ take $13.\dot{9}643\dot{2}$. *Diff.* $71.\dot{8}619\dot{3}$.
6. From $10.0\dot{4}1\dot{3}$ take $.2\dot{6}\dot{4}$. *Diff.* $9.\dot{7}76694\dot{8}$.

---

[m] The reason of this rule will be plain from the preceding note.

256. MULTIPLICATION OF CIRCULATING DECIMALS.

RULE I. Turn both terms into their equivalent vulgar fractions, and multiply those fractions together.

II. Reduce the product to its equivalent decimal, and let the work be continued until the decimal figures repeat; then mark the first and last repeating figures, and it will be the product required[a].

EXAMPLES.

1. Multiply $.\dot{1}\dot{2}$ by $.3\dot{4}$.

OPERATION.

$$.\dot{1}\dot{2} = \frac{\dot{1}\dot{2}}{99} = \frac{4}{33}. \quad .3\dot{4} = \frac{31}{90}.$$

$$\textit{Then } \frac{4}{33} \times \frac{31}{90} = \frac{124}{2970} = \frac{62}{1485} = .0\dot{4}17508\dot{8} \textit{ the product.}$$

Explanation.

I first turn the decimals into their equivalent vulgar fractions; I then multiply the fractions together, and reduce the product to a decimal, continuing the quotient until the figures recur.

2. Multiply $12.\dot{3}$ and $.\dot{4}5\dot{6}$ together.

$$12.\dot{3} = 12\frac{3}{9} = \frac{111}{9} = \frac{37}{3}. \quad .\dot{4}5\dot{6} = \frac{456}{999} = \frac{152}{333}$$

$$\frac{37}{3} \times \frac{152}{333} = \frac{5624}{999} = 5.\dot{6}2\dot{9} \textit{ product.}$$

3. Multiply $.2\dot{5}$ by $\dot{3}\dot{6}$. *Prod.* $.0\dot{9}2\dot{9}$.
4. Multiply $37.2\dot{3}$ by $.2\dot{6}$. *Prod.* $9.92\dot{8}$.
5. Multiply $.\dot{3}$, $1.\dot{2}$, and $.6\dot{7}$ together. *Prod.* .27613168 &c.
6. Multiply $.\dot{5}\dot{2}$, $3.\dot{1}$ and $.\dot{2}3\dot{5}$ together. *Prod.* .38440235 &c.
7. Multiply $.\dot{2}3$, $2.\dot{3}\dot{4}$, and $1.\dot{2}$ together. *Prod.* 1.14435839 &c.
8. Multiply $.\dot{4}$, .4, and $.000\dot{4}$ together. *Prod.* $.0000\dot{7}9012345\dot{6}$.

---

[a] The reason of this operation will be plain; for if two quantities be respectively equal to other two, the product of the first two will equal the product of the last two; wherefore, in the present case, the product of the given decimals is evidently found when the product of their equivalent vulgar fractions is found.

257. DIVISION OF CIRCULATING DECIMALS.

RULE I. Change the divisor and dividend into their equivalent vulgar fractions, (Art. 249, 250, or 251.) and divide the latter by the former, by Art. 204.

II. Reduce the quotient to its equivalent decimal, and it will be the answer*.

EXAMPLES.

1. Divide $.\dot{5}\dot{4}$ by $.\dot{3}$.

OPERATION.

$$\textit{First } .\dot{5}\dot{4} = \frac{54}{99} = \frac{6}{11}$$

$$\textit{And } .\dot{3} = \frac{3}{9} = \frac{1}{3}$$

$$\textit{Then } \frac{6}{11} \times \frac{3}{1} = \frac{18}{11} = 1.\dot{6}\dot{3} \textit{ quotient.}$$

Explanation.

I First reduce both repetends to the equal vulgar fractions $\frac{6}{11}$ and $\frac{1}{3}$; then, after inverting the divisor $\frac{1}{3}$, I multiply both fractions together, and the result $\frac{18}{11}$ is next reduced to a decimal, which is the quotient.

2. Divide $.4\dot{8}$ by $1.\dot{2}\dot{3}$.

$$\textit{Thus } .4\dot{8} = \frac{44}{90} = \frac{22}{45} \qquad 1.\dot{2}\dot{3} = 1\frac{23}{99} = \frac{122}{99}.$$

$$\textit{Then } \frac{\overset{11}{\cancel{22}}}{\underset{5}{\cancel{45}}} \times \frac{\overset{11}{\cancel{99}}}{\underset{61}{\cancel{122}}} = \frac{121}{305} = .3967213 \textit{ quotient.}$$

3. Divide $.\dot{2}\dot{4}$ by $.\dot{7}$. *Quot.* $.\dot{3}1168\dot{8}$

4. Divide $234.\dot{6}$ by $1.\dot{3}$. *Quot.* 176.

5. Divide $9.92\dot{8}$ by $.2\dot{6}$. *Quot.* $37.2\dot{3}$.

6. Divide $13.5\dot{1}6953\dot{3}$ by $4.\dot{2}9\dot{7}$. *Quot.* $3.\dot{1}4\dot{5}$.

7. Divide $12.\dot{3}45\dot{6}$ by $.00\dot{8}\dot{1}$. *Quot.* $1508.\dot{9}\dot{1}$.

8. Divide $.\dot{3}\dot{6}$ by $\dot{2}5$. *Quot.* $1.\dot{4}229249001185770750988\dot{1}$.

---

* What has been said of the product (in the foregoing note) is equally true of the quotient, as is sufficiently evident from Art. 204.

DUODECIMALS.

258. If an unit be divided into 12 equal parts, each of these parts into 12, each of these latter into 12, and so on without end; such fractions are called *Duodecimal Fractions.*

259. The parts into which the unit is divided are called *primes,* and marked thus $'$; those into which the prime is divided, are called *seconds,* and are marked thus $''$; those into which the second is divided are called *thirds,* and marked thus $'''$; and so on for the succeeding divisions, viz. *fourths* ^{iv}; *fifths* ^{v}; *sixths* ^{vi}; &c.

260. Duodecimals [p], or Cross Multiplication, is a method of finding the content of any rectangular surface, the length and breadth being given in feet, inches, and duodecimal parts, and is employed by artificers in computing their work.

RULE I. Under the multiplicand write the multiplier, so that feet may stand under feet, primes under primes, &c.

II. Multiply each term in the multiplicand (beginning at the lowest) by the feet in the multiplier, and write the result of each under its respective term; that is, carry one for every 12 that arises, and set down the remainder.

III. Multiply in the same manner by the primes, and let the result of each term stand one place to the right of that term in the multiplicand.

IV. Multiply in like manner by the seconds, and set each re-

---

[p] The name is derived from the Latin *duodecim,* twelve, and is expressive of the nature of the division and subdivision of unity, which take place in these operations. The term *Cross Multiplication* arises from the *cross* method of operating, or multiplying *cross-ways.* This rule is much in use among Artificers, as it supplies them with a ready method of determining the dimensions of their work and materials.

Brick-layers, masons, glaziers, and others, measure their work by the square foot; painters, paviors, plasterers, &c. by the square yard; tiling, slating, and flooring, are usually measured by the square of 100 feet, and brick-work is frequently measured by the rod of $16\frac{1}{2}$ feet.

sult two places to the right: by the thirds, and set each result three places to the right; and so on to the end.

V. Add all the products together, (observing continually to carry one for every 12, and to set down the remainder,) the sum will be the answer[q].

[q] This rule may be proved by vulgar fractions; thus, ex. 3. 5f. $8' = 5 + \frac{8}{12}$, and 2f. $5' = 2 + \frac{5}{12}$, wherefore $\overline{5 + \frac{8}{12}} \times \overline{2 + \frac{5}{12}} = 13 + \frac{8}{12} + \frac{4}{12.12}$ or 13f. 8′ 4″, as in the example; and the same may be shewn in all cases.

It will be useful to remember the following particulars; namely, that

Feet multiplied into feet produce feet.
Feet multiplied into inches produce inches.
Feet multiplied into seconds produce seconds.
&c. &c.
Inches multiplied into inches produce seconds.
Inches multiplied into seconds produce thirds.
&c. &c.
Seconds multiplied into seconds produce fourths.
Seconds multiplied into thirds produce fifths.
&c. &c.

And in general, the product of any two terms will be of that denomination, which is denoted by the sum of the numbers which express the denominations of those terms.

Thus seconds multiplied by thirds produce fifths, for $2 + 3 = 5$; and the same universally.

EXAMPLES.

1. [r] Multiply 2*f.* 3′ 1″ by 5*f.* 7′ 9″.

OPERATION.

f.	′	″			
2	3	1	*Multiplicand.*		
5	7	9	*Multiplier.*		
11	3	5			
1	3	9	7‴		
	1	8	3	9iv	
12*f.*	8′	10″	10‴	9iv	*Prod.*

Explanation.

I first multiply the multiplicand by 5, carrying 1 for every 12, and putting down the remainders under their like for the first line. I next multiply by the 7 primes, putting the results one place to the right of the last, and carrying as before for the second line. I then multiply by the 9 seconds, putting the results two places to the right, and carrying as before for the third line. I lastly add these three lines together (carrying as before) for the product.

2. Multiply 4*f.* 3′ 2″ 1‴
By 1*f.* 2′ 3″ 4‴

	4	3	2	1			
		8	6	4	2iv.		
		1	0	9	6	3v.	
			1	5	0	8	4vi.
Product	5	0	10	7	8	11	4

3. Multiply 5*f.* 8′ by 2*f.* 5′. *Prod.* 13*f.* 8′ 4″.
4. Multiply 3*f.* 6′ by 7*f.* 9′. *Prod.* 27*f.* 1′ 6″.
5. Multiply 7*f.* 5′ 9″ and 3*f.* 5′ 3″ together. *Prod.* 25*f.* 8′ 6″ 2‴ 3iv.

---

[r] These examples may be proved by Practice, and by Decimals; thus, Ex. I.

By Practice.

6′	$\frac{1}{2}$	2*f.*	3′	1″		
				5		
		11	3	5		
1	$\frac{1}{6}$	1	1	6	6	
6″	$\frac{1}{2}$		2	3	1	
3	$\frac{1}{2}$		1	1	6	6
				6	9	3
		12	8	10	10	9

By Decimals.

```
 2.2569 &c. = 2f. 3′ 1″
 5.6458 &c. = 5   7  9
     180552
    112845
    90276
   135414
  112845
  12.74200602
           12
   8.90407224
           12
  10.84886688
           12
  10.18640256
           12
   2.23683072
```

It may be remarked, that the operation by *practice* gives the exact answer, while that by decimals is 7 fourths (or $\frac{7}{20736}$ of a square foot) short of the truth, in consequence of both factors being infinite, and too few figures taken for the operation: if the factors had been continued, they would have been respectively $2.256941\dot{6}$, and $5.6459\dot{3}$; whence (by proceeding according to articles 256 and 251.) the exact answer would have been obtained.

6. Multiply 10*f.* 4′ 5″ and 7*f.* 8′ 6″ together. *Prod.* 79*f.* 11′ 0″ 6‴ 6iv.

7. Multiply 5*f.* 4′ 8″ by 9*f.* 8′ 6″. *Prod.* 52*f.* 3′ 9″ 8‴.

8. Multiply 311*f.* 7′ 5″ by 36*f.* 4′ 11″. *Prod.* 11345*f.* 11′ 1″ 5‴ 7iv.

9. What will a mahogany side-board cost, which is 5 feet 11 inches long, and 3 feet 7 inches wide throughout, at 4*s.* 6*d.* per square foot? *Ans.* 4*l.* 15*s.* 4*d.* $\frac{1}{4}$.

10. What sum will pay for new glazing a hall window containing 60 squares, each 1*f.* 2′ 3″ long, and 11′ 5″ wide, at 3*s.* 8*d.* per square foot? *Ans.* 12*l.* 8*s.* 6*d.* $\frac{1}{4}$.

11. What is the price of a marble slab, the length of which is 5 feet 7 inches, and breadth 3 feet 8 inches, at 9*s.* per square foot? *Ans.* 9*l.* 4*s.* 3*d.*

12. A room which is 66 feet 10 inches about, was wainscotted 3 feet 11 inches upwards from the floor; what did it come to at 2*s.* 7*d.* $\frac{1}{2}$. per square foot? *Ans.* 34*l.* 7*s.* 1*d.* $\frac{1}{2}$.

13. A drawing-room which is 32 feet 8 inches long, and 25 feet 9 inches wide, is surrounded with a cornice 3$\frac{1}{2}$ inches wide, the gilding of which cost 3*l.* 5*s.* 6*d.* required what sum was charged per square foot? *Ans.* 1*s.* 11*d.* $\frac{323}{1207}$.

14. The paving of a brew-house, 24 feet 11 inches long, and 34 feet 6 inches broad, cost 7*s.* 9*d.* per square yard; what did the whole amount to? *Ans.* 37*l.* 0*s.* 2*d.*

15. The expense of digging, planting, and manuring a kitchen-garden amounted to 14*l.* 1*s.* 8*d.* how much is that per square yard, supposing the length to be 109 feet 6 inches, and the breadth 58 feet 6 inches? *Ans.* 4*d.* $\frac{3}{4}$.

16. What sum must I pay for painting a room 48 feet 10 inches about, and 9 feet 10 inches high, at 2*s.* 8*d.* $\frac{1}{4}$ per square yard? *Ans.* 7*l.* 5*s.* 7*d.* $\frac{1}{4}$.

INVOLUTION*.

261. The power of any number is the product that arises by multiplying that number into itself; and the product (if necessary) into the given number; and this product (if necessary) into the given number, and so on continually.

262. The product arising from *one* multiplication, is called the square, or second power of the given number; the product arising from *two* successive multiplications, is called the cube, or third power; the product of *three* multiplications, the fourth power; of *four* multiplications, the fifth power, and so on.

263. The power of any number is denoted by a small figure, called the *index* or *exponent* of the power, placed over, and a little to the right of, the given number.

Thus 3^2 denotes the *second power*, or *square* of 3, the small 2 being the *index* or *exponent* of the second power; 7^3 denotes the *third power* of 7, where 3 is the index; $\overline{21}^5$ denotes the *fifth* power of 21, where 5 is the index, &c.

264. Involution teaches to find the powers of any given number.

265. *To involve whole numbers or decimals to any power.*

RULE I. Multiply the given number into itself for the square, and this product into the given number for the cube, and so on continually for the higher powers; observing, that to obtain any power, the number of successive multiplications will always be one less than the index of the required power.

II. If there are decimals in the number given to be involved, mark off the decimals in each product, according to the rule for multiplying decimals, Art. 224.

---

* The name Involution is derived from the Latin *involvo*, to wrap or fold in. The number to be involved is called *the root* of the proposed power; the number arising from the involution is called *the power* of the given root. The terms *square* and *cube* are applied to certain numbers, because they arise by processes similar to the known method of computing the capacity of those figures: and because the second power is called a *square*, and the third power a *cube*, the second root is named the *square root*, and the third root the *cube root*; the fourth power is sometimes called the *biquadrate*, (*bis quadratus*,) and the fourth root the *biquadrate root*. Particular names for other powers and roots are to be found in old books, but they are now seldom used; see the note on Art. 52. part 3.

EXAMPLES.

1. What is the fourth power of 12?

OPERATION.

12 = 1st *power.*
12
144 = 2nd *power.*
12
1728 = 3d *power.*
12
20736 = 4th *power.*

Explanation.

Here the index of the required power being 4, three multiplications are necessary: the first produces the *square*, or *second power*; the second produces the *cube*, or *third power*; and the third produces the *biquadrate*, or *fourth power*, as was required.

2. Involve 2.3 and .103 each to the fifth power.

2.3	= 1*st power or root* =	.103
2.3		.103
69		309
46		103
5.29	= 2*d power* =	.010609
2.3		.103
1587		31827
1058		10609
12.167	= 3*d power* =	.001092727
2.3		.103
36501		3278181
24334		1092727
27.9841	= 4*th power* =	.000112550881
2.3		.103
839523		337652643
559682		112550881
64.36343	= 5*th power* =	.000011592740743

3. Involve 234 to the square. *Square* 54756.
4. What is the cube of 54? *Cube* 157464.
5. Involve 100.2 to the third power. *Third power* 1006012.008.
6. Involve 94.75 to the fourth power. *Fourth power* 80596628.44140625.

266. *To involve a simple fraction to any power.*

RULE. Involve the numerator and denominator each separately to the given power, and the results will be the respective terms of a new fraction, which will be the power required.

7. Involve $\frac{2}{3}$ to the square, and $\frac{4}{5}$ to the cube.

Thus $\overline{\frac{2}{3}}\big|^2 = \frac{2 \times 2}{3 \times 3} = \frac{4}{9}$, *which is the square of* $\frac{2}{3}$.

And $\overline{\frac{4}{5}}\big|^3 = \frac{4 \times 4 \times 4}{5 \times 5 \times 5} = \frac{64}{125}$, the cube of $\frac{4}{5}$.

8. Involve $\frac{1}{2}$ to the square. *Square* $\frac{1}{4}$.

9. Involve $\frac{5}{6}$ to the cube. *Cube* $\frac{125}{216}$.

10. Required the biquadrate of $\frac{3}{4}$. *Biquadrate* $\frac{81}{256}$.

267. *To involve a mixed number to any power.*

RULE I. Either reduce the given mixed number to an improper fraction, by Art. 172; involve both terms of this fraction by the last rule; reduce the resulting improper fraction to its proper terms, by Art. 173, and the result will be the power. Or,

II. Reduce the fractional part of the given number to a decimal, Art. 233. subjoin this to the whole number, and involve the result to the given power, by Art. 265.

11. Involve $2\frac{3}{4}$ to the second power.

Thus, Art. 172. $2\frac{3}{4} = \frac{2 \times 4 + 3}{4} = \frac{11}{4}$.

Then $\overline{\frac{11}{4}}\big|^2 = \frac{11 \times 11}{4 \times 4} = \frac{121}{16} =$ (Art. 173.) $7\frac{9}{16}$ *the power required, or,*

Secondly, $2\frac{3}{4} = 2.75$ *by Art.* 233.

Then $\overline{2.75}\big|^2 = 2.75 \times 2.75 = 7.5625$, *the power, as before; for this decimal* .5625 *being reduced to a vulgar fraction,* (*Art.* 232.) *will* $= \frac{9}{16}$ *as above.*

12. Involve $\frac{2\frac{1}{3}}{4\frac{2}{5}}$ to the cube.

Thus, Art. 178. $\frac{2\frac{1}{3}}{4\frac{2}{5}} = \frac{35}{66}$.

Then $\overline{\frac{35}{66}}\big|^3 = \frac{42875}{287496} =$ *the cube required.*

13. Required the square of $1\frac{1}{2}$. *Square* $2\frac{1}{4}$.

14. What is the cube of $2\frac{3}{4}$? *Cube* $20\frac{51}{64}$.

15. To find the biquadrate of $10\frac{1}{3}$. *Biquad.* $11401\frac{40}{81}$.

EVOLUTION[t].

268. The root of any number is that which being multiplied once, or oftener continually into itself, will produce the said number.

269. A number which being multiplied *once* into itself produces the given number, is called the square root of that number; a number which multiplied successively *twice*, produces the given number, is called the cube root of that number; if it produces the given number by three successive multiplications, it is called the fourth, or biquadrate root of that number, and so on.

270. The root of any number is denoted either by a radical sign $\surd$, with a small figure expressive of the number of the root

---

[t] The name *Evolution* is derived from the Latin *evolvo*, to unfold. With respect to the operation, the evolution of roots consisting of only one figure is merely a simple mechanical process, the reason of which immediately appears: but when the root consists of several figures, the grounds of the rule by which it is extracted are by no means obvious. A respectable writer, whose name would do honour to these pages, observes, that "any person who can extract the "square and cube root in Algebra, will not be at a loss *to demonstrate* the "rules of square and cube root" in Arithmetic; "and to those who cannot, "a demonstration would be of little or no use." The truth is, that the common rules for the extraction of roots, either in Algebra or Arithmetic, as far as I have been able to learn, *have never yet been demonstrated* independently, and without supposing that both the root and its power are previously known: having the root given, its power, although unknown, is easily obtained by multiplication; but the root being unknown, cannot be obtained from the power by the converse operation of division, because the divisor is not known: hence it appears, that the rules for Evolution were first discovered mechanically, or by dint of trial; and the only proof that they are true is, that the number arising from their operation, being *involved*, produces the given power. *See the note on Art.* 57. *part* 3. Mr. Wood has shewn very clearly in the extraction of the cube root, how the several steps in the Arithmetical and Algebraic operations respectively coincide with each other. *Elem. of Algeb.* pp. 62, 63. *Third Edit.*

The square root of any number, is a mean proportional between unity and that number; the cube root is the first of two mean proportionals between unity and the given number; the biquadrate root is the first of three mean proportionals between unity and the given number; and in general, if between unity and any power there be taken mean proportionals in number one less than the index of that power, the first of these will be the root required.

over it, and the whole placed before the given number; or by a fractional index or exponent, placed over, and a little to the right of the given number.

Thus $\sqrt{3}$ or $3^{\frac{1}{2}}$ denotes the square root of 3; $\sqrt[3]{7}$ or $7^{\frac{1}{3}}$ denotes the cube root of 7; $\sqrt[5]{21}$ or $\overline{21}|^{\frac{1}{5}}$ denotes the fifth root of 21, &c.

271. Evolution teaches to find the roots of any given number.

Thus to extract the square root of a number, is to find such a number which being multiplied once into itself, produces the given number: to extract the cube root, is to find a number which being multiplied into itself, and that product into the same number, produces the given number; and so for other roots.

EXTRACTION OF THE SQUARE ROOT.

272. The following table contains the first nine whole numbers which are exact squares, with the square root of each placed under its respective number.

SQUARES.......	1.	4.	9.	16.	25.	36.	49.	64.	81.
SQUARE ROOTS.	1.	2.	3.	4.	5.	6.	7.	8.	9.

From this table the root of any exact square, being a single figure, may be obtained by inspection, as is plain.

273. *To extract the square root, when it consists of two or more figures, from any number.*

RULE I. Make a point over the units' place of the given number, another over the hundreds, and so on, putting a point over every second figure; whereby the given number will be divided into periods of two figures each, except the left hand period, which will be either one or two, according as the number of figures in the whole is odd or even.

II. Find in the table the greatest square number not greater than the left hand period, set it under that period, and its root in the quotient.

III. Subtract the said square from the figures above it, and to the remainder bring down the next period for a dividend.

IV. Double the root, (or quotient figure,) and place it for a divisor on the left of the dividend.

V. Find how often the divisor is contained in the dividend, omitting the place of units, and place the number (denoting

how many) both in the quotient, and on the right of the divisor.

VI. Multiply the divisor (thus augmented) by the figure last put in the quotient, and set the product under the dividend.

VII. Subtract, and bring down the next period to the remainder for a dividend; and to the left of this bring down the last divisor with its right hand figure doubled, for a divisor.

VIII. Find how often the divisor is contained in the dividend, omitting the units as before; put the number denoting how often in the quotient, and also on the right of the divisor. Multiply, subtract, bring down the next period, and also the divisor with its right hand figure doubled, &c. as before, and proceed in this manner till the work is finished.

IX. If there is a remainder, periods of ciphers may be successively brought down, and the work continued as before, observing that the quotient figures which arise will be decimals; and if there be an odd decimal figure in the given number, a cipher must be subjoined, to make the right hand period complete. Also for every period of superfluous ciphers, either on the left or right of the given number, a cipher must be placed in the quotient. The operations may be proved by involving the root to the square, (Art. 265.) and adding in the remainder, if any.

EXAMPLES.

1. Extract the square root of 54756.

OPERATION.

```
     54756(234 = root
     4
  43)147
     129
 464) 1856
      1856
```

Proof.

```
  234 = root
  234 = root
  936
 702
468
54756 = square.
```

Explanation.

I first place a point over the units, then over the hundreds, then over the ten thousands; 5 being the first period, I find from the table the greatest square 4, contained in it; this 4 I place under the 5, and its root 2 in the quotient, and having subtracted, I bring down to the remainder 1 the next period 47, making 147 for the dividend; I double the quotient figure 2, and place the double, viz. 4, for a divisor, to the left. Omitting the units 7, I ask how often 4 is contained in 14, and find it goes 3 times; this 3 I put both in the quotient and divisor, making the latter 43; this I multiply by the quotient figure 3, and subtract the product 129 from the dividend. To the remainder 18 I bring down the next period 56, making the new dividend 1856; to the left of this I bring down the divisor with its last figure 3 doubled, making 46: I then ask how often 46 goes in 185, (omitting the 6,) it goes 4 times, I therefore put the 4 both in the quotient and on the right of the divisor, and multiply as before: there being neither a remainder, nor any more figures to bring down, the operation is finished.

2. Extract the square root of .000064807.

OPERATION*.

```
.0000648070).008050279 root.
     64
1605)   8070
        8025
161002)   450000
          322004
1610047)  12799600
          11270329
16100549)  152927100
           144904941
Remainder    8022159
```

Explanation.

Here, in order to complete the right hand period, I subjoin a cipher; and there being 2 periods of ciphers to the left, I prefix a cipher for each period to the root. In the second step, having brought down the 80, I find that 16 will not go in 8; I therefore put a cipher both in the quotient and divisor, and then bring down the next period 70; and the like in the next step. I bring down a period of ciphers both there and in each following step.

3. Extract the square root of 95.801234.

```
          95.801234(9.78781 root.
          81
     187)1480
         1309
    1948)  17112
           15584
   19567)  152834
           136969
  195748)  1586500
           1565984
 1957561)  2051600
           1957561
Remainder    94039
```

4. Required the square root of 529. *Root* 23.

5. To find the square root of 104976. *Root* 324.

6. What is the square root of 298116? *Root* 546.

* These operations may be proved three ways. First, by involving the root to the square as in ex. 1. and adding the remainder to the square: the result, if the work be right, will equal the given number. Secondly, by casting out the nines: thus, cast out the nines from the root, and multiply the excess into itself; cast the nines out of the product, reserving the excess; cast the nines out of the remainder, subtract the excess from the dividend, and cast the nines out of what remains: if this excess equals the former, the work may (with the restriction mentioned in the note on Art. 41.) be presumed to be right. Thirdly, by addition, similar to the proof of long division, Art. 41.

7. Extract the square root of 974169. *Root* 987.
8. What is the square root of 1046529? *Root* 1023.
9. Extract the square root of 867.8916. *Root* 29.46.
10. Required the square root of 32.72869681. *Root* 5.7209.
11. Find the square root of 70. *Root* 8.3666, &c.
12. What is the square root of .000294? *Root* .0171464, &c.
13. What is the square root of 989? *Root* 31.44837, &c.
14. Find the square root of 6.27. *Root* 2.503996805, &c.
15. Required $\sqrt{.0001524157875019052 1}$. *Root* .0123456789.

274. *To extract the square root of a vulgar fraction, both terms of which are exact squares.*

RULE. Extract the root of the numerator, and likewise of the denominator; these two roots will be respectively the terms of a new fraction, which will be the root required.

16. Extract the square root of $\frac{4}{9}$.

OPERATION.

$\sqrt{\frac{4}{9}} = \frac{2}{3}$ *the root required.*

Explanation.

Here the root of 4 is 2, and the root of 9 is 3, therefore $\frac{2}{3}$ will be the root of the given fraction.

17. What are the square roots of $\frac{9}{16}$, $\frac{25}{144}$, $\frac{36}{169}$, and $\frac{1681}{76176}$? *Roots* $\frac{3}{4}$, $\frac{5}{12}$, $\frac{6}{13}$, and $\frac{41}{276}$.

275. *To extract the square root of a vulgar fraction, the terms of which are not both squares.*

RULE. Reduce the given fraction to a decimal, (Art. 233.) and extract the root of this decimal for the answer.

18. What is the square root of $\frac{1}{2}$?

First $\frac{1}{2} = .5$ *by Art.* 233. *Then* $\sqrt{.5} = .70710678119$, &c. (*Art.* 273.) *the root required.*

19. To find the square root of $\frac{3}{4}$. *Root* .8660254, &c.

20. What is the square root of $\frac{7}{8}$? *Root* .935403989, &c.

276. *To extract the square root of a mixed number.*

RULE. Reduce the fraction to a decimal, (Art. 233.) to which prefix the whole number, extract the square root of the result by Art. 273. and it will be the root required.

21. To find the square root of $8\frac{3}{4}$.

First (*Art.* 233.) $8\frac{3}{4} = 8.75$.

Then (*Art.* 273.) $\sqrt{8.75} = 2.95803989$, *the root required.*

22. What is the square root of $\frac{1\frac{2}{3}}{4\frac{5}{6}}$?

Thus (*Art.* 178.) $\frac{1\frac{2}{3}}{4\frac{5}{6}} = \frac{30}{87} = \frac{10}{29} = .344827586206$, &c.

Therefore $\sqrt{.344827586206}$, &c. $= .5872202$, &c. (*Art.* 273.) $=$ *the root required.*

23. Required the square root of $1\frac{1}{2}$. *Root* 1.22474487, &c.

24. What is the square root of $7\frac{4}{5}$? *Root* 2.792848, &c.

25. What is the square root of $\frac{2\frac{1}{2}}{3\frac{3}{4}}$? *Root* .8164965, &c.

277. Sometimes it happens, that the given mixed number being reduced to its equivalent improper fraction, both the terms will be rational. In this case it will be best to extract the roots of the numerator and denominator separately, and they will form an improper fraction, which must be reduced to its proper terms, (Art. 173.)

26. Extract the square root of $2\frac{1}{4}$.

Thus (*Art.* 172.) $2\frac{1}{4} = \frac{9}{4}$. *Then* (*Art.* 274.) $\sqrt{\frac{9}{4}} = \frac{3}{2}$ $= 1\frac{1}{2}$ (Art. 173.) *the root required.*

27. What is the square root of $2\frac{7}{9}$? *Root* $1\frac{2}{3}$.

28. What is the square root of $30\frac{1}{4}$? *Root* $5\frac{1}{2}$.

29. What is the square root of $11\frac{1}{9}$? *Root* $3\frac{1}{3}$.

30. Required the square root of $156\frac{1}{4}$. *Root* $12\frac{1}{2}$.

278. PROMISCUOUS EXAMPLES FOR PRACTICE.

1. The side of a square kitchen garden is 63 yards; how many square yards does the garden contain? *Ans.* 3969 *square yards.*

2. An army consisting of 87616 men, is to be arranged in the form of a square; how many will a side contain? *Ans.* 296 *men.*

3. What will the paling of a square garden consisting of 2209 square yards cost, at 3*s.* 6*d.* per yard? *Ans.* 32*l.* 18*s.*

4. Required the length of a ladder, the foot of which being pitched 12 feet from the wall, its top will reach a window 18 feet from the ground[x]? *Ans.* 21 *feet,* 7.59969, *&c. inches.*

5. A rope 120 fathoms long is extended from the top of a cliff, to a boat moored at 80 fathoms from its base; required the perpendicular height of the cliff[y]? *Ans.* 89.4427, *&c. fathoms.*

6. Required a mean proportional between 36 and 2401[z]? *Ans.* 294.

7. A rectangular field has its sides equal to 210 and 300 yards respectively; what length must the side of a square be to contain an equal area? *Ans.* 250.9979, *&c. yards.*

8. The diagonal (or straight line joining the opposite corners) of a chess-board, measures 30 inches; required the length of the side? *Ans.* 21.213203, *&c. inches.*

9. A wall is supported 13 feet from the ground by a shoar 16 feet long; how far is the foot of the shoar distant from the base of the building? *Ans.* 9 *feet,* 3.9285, *&c. inches.*

10. A gentleman has a table 5 feet wide, and 14 feet long, and wants three others to be made, each square, and all together of equal dimensions with the former; required the side of each? *Ans.* 4 *feet,* 9.9654, *&c. inches.*

EXTRACTION OF THE CUBE-ROOT.

279. The following table contains the first nine whole num-

---

[x] The length of the ladder, the perpendicular distance of its top from the ground, and the distance of its foot from the wall, together form a right angled triangle; and it is demonstrated in the 47th proposition of the first book of Euclid's Elements, that the square of the longest side of such triangle is equal to the sum of the squares of the two remaining sides; wherefore in the present instance, $\sqrt{\overline{12}|^2 + \overline{18}|^2} = 21.633$, &c. feet, $=$ the length required.

[y] From the foregoing note it follows, that the difference of the squares of the longest and of either of the remaining sides, is equal to the square of the other side; whence $\sqrt{\overline{120}|^2 - \overline{80}|^2} = 89.44$, &c. fathoms.

[z] The square root of the product of any two numbers is a mean proportional between them.

bers which are exact cubes, together with their cube roots, where each root is placed exactly under the number of which it is the root, whereby the cube root of any cube number within its limits may be found.

CUBES	1.	8.	27.	64.	125.	216.	343.	512.	729.
CUBE-ROOTS.	1.	2.	3.	4.	5.	6.	7.	8.	9.

280. *To extract the cube root, consisting of several figures, from any number.*

Rule I. Put a point over the units' place, and also one over every third figure, counting from the units; whereby the given number will be divided into periods of three figures each, except the left hand period, which may be either one, two, or three figures.

II. Find by the table the greatest cube in the left hand period, set the said cube under that period, and its root in the quotient.

III. Subtract the cube from the period above it, and to the remainder bring down the next period for a dividend.

IV. Multiply the square of the root by 300, and place the product to the left of the dividend for a divisor.

V. Find how often the divisor is contained in the dividend, and place the number in the quotient for the next figure of the root.

VI. Multiply the divisor by the last figure of the root. Multiply all the figures in the root, except the last, by 30, and that product by the square of the last. Cube the last figure of the root. Add these three together, and call their sum the *subtrahend.*

VII. Subtract the subtrahend from the dividend, and bring down the next period for a new dividend.

VIII. Find a new divisor by proceeding as before, viz. multiplying the square of the whole of the root found by 300. Divide. Find a new subtrahend as before, and proceed in this manner until the work is finished.

IX. Decimals must likewise be pointed over every third figure from the units' place, and if there are not decimals enough to complete the right hand period, the deficiency must be supplied by ciphers. If the given number consists of whole

numbers and decimals, the root will consist of as many places of whole numbers, as there are periods of whole numbers; and as many decimals, as there are periods of decimals. If there is a remainder after all the figures are brought down, the work may be continued, by bringing down periods of ciphers [a].

EXAMPLES.

1. Extract the cube root of 32768.

OPERATION.

$$
\begin{array}{ll}
 & 3\dot{2}76\dot{8}(32 \textit{ root.} \\
 & 27 \\
\overline{3}|^2 \times 300 = 2700) & \overline{5768} \textit{ dividend.} \\
2700 \times 2 = & 5400 \\
3 \times 30 \times \overline{2}|^2 = & 360 \\
\overline{2}|^3 = & 8 \\
 & \overline{576} \textit{ subtrahend.}
\end{array}
$$

Explanation.

Under the first period 32, I place the greatest cube contained in it, viz. 27; and the root 3 in the quotient; I then subtract 27 from 32, and to the remainder 5 bring down 768 for the dividend: to the left of which I place the divisor, viz. the square of 3 (or 9) multiplied by 300, which gives 2700: this number I find goes *twice* in the dividend, I therefore put 2 in the quotient. I multiply the divisor by this quotient figure, and place the product 5400 under the dividend; I multiply all the figures in the root, (viz. 3.) except the last, by 30, and that product by $\overline{2}|^2$ the square of the last, which gives 360. I then cube the last figure, viz. 2, which gives 8: these three I add together for the subtrahend, which being the same as the dividend, and all the periods being brought down, the work is finished.

---

[a] When the root of a decimal *only*, having several places of ciphers on its left, is to be extracted, for every complete period of ciphers, observe to prefix a cipher to the root.

This rule is proved by involving the root found to the cube, and adding in the remainder (if any) to the last line of the work; the sum will be equal to the given number, if the operation is right.

2. What is the cube root of 99.252847?

OPERATION.

$\dot{9}9.25\dot{2}84\dot{7}(4.63$

64

$\overline{4}|^2 \times 300 = 4800)\overline{35252}$ *dividend.*

$4800 \times 6 = \quad 28800$

$4 \times 30 \times \overline{6}|^2 = \quad 4320$

$\overline{6}|^3 = \quad 216$

33336 *subtrahend.*

$\overline{46}|^2 \times 300 = 634800)\overline{1916847}$ *second divid.*

$634800 \times 3 = \quad 1904400$

$46 \times 30 \times \overline{3}|^2 = \quad 12420$

$\overline{3}|^3 = \quad 27$

1916847 *second subtr.*

3. Required the cube root of 12167. *Root* 23.
4. Required the cube root of 157464. *Root* 54.
5. Extract the cube root of 91125. *Root* 45.
6. What is the cube root of 14886936? *Root* 246.
7. What is the cube root of 43614208? *Root* 352.
8. Extract the cube root of 128.024064. *Root* 5.04.
9. Required the cube root of 1879080.904. *Root* 123.4.
10. Extract the cube root of 1.006012008. *Root* 1.002.
11. Required the cube root of 27407.028375. *Root* 30.15.
12. What is the cube root of .0001357? *Root* .05138, &c.
13. What is the cube root of 2? *Root* 1.2599, &c.

281. *To extract the cube root of a vulgar fraction, or mixed number.*

RULE. If the terms of the fraction be both rational, extract the root of the numerator for a numerator, and of the denominator for a denominator (Art. 280.); but if they are not both rational, reduce the fraction to a decimal, (Art. 233.) and extract the root of the latter[b]. (Art. 280.)

For a mixed number, reduce the fractional part to a decimal,

---

[b] The former part of the rule is the most convenient, when it can be applied; the latter part is general, and applies equally, whether the terms of the given fraction be rational or irrational. The operations are proved by involution, as in the preceding rule.

(Art. 283.) to which prefix the whole number, and extract the root as before.

14. Extract the cube root of $\frac{81}{192}$.

This fraction reduced to its lowest terms is $\frac{81}{192} = \frac{27}{64}$, *the terms of which are both cubes.*

Therefore $\sqrt[3]{\frac{27}{64}} = \frac{3}{4}$, *the root required.*

15. Extract the cube root of $\frac{1}{2}$.

Thus $\frac{1}{2} = .5$, *then* $\sqrt[3]{.5} = .7938$, &c. *the root.*

16. Required the cube root of $3\frac{1}{2}$.

Thus $\sqrt[3]{3\frac{1}{2}} = \sqrt[3]{3.5} = 1{,}518$, &c. *the root.*

17. Required the cube root of $\frac{27}{125}$. *Root* $\frac{3}{5}$.

18. Required the cube root of $\frac{512}{2744}$. *Root* $\frac{4}{7}$.

19. Required the cube root of $\frac{4}{7}$. *Root* .829, &c.

20. What is the cube root of $8\frac{5}{7}$? *Root* 2.0564, &c.

282. PROMISCUOUS EXAMPLES FOR PRACTICE.

1. What is the length, breadth, and height, of a cubical room, which contains 2197 cubic feet of air? *Ans.* 13 *feet.*

2. Required the side of a cubical box, which will hold 2744 cubic inches of flour. *Ans.* 14 *inches.*

3. A cubical box holds 9261 cubic inches of corn, how many square feet of deal are there in it? *Ans.* 18 *feet,* 54 *inches.*

4. A cubical cistern contains 125 cubic feet of water, what is the value of the lead, at $2d.\frac{1}{2}$ per lb. allowing $4\frac{1}{2}$ lb. to every square foot, and 12 square feet for the rim? *Ans.* 6*l.* 15*s.* $6d.\frac{1}{4}$ .5.

5. The solid content of the earth is estimated at 265404598080 cubic miles; required the side of a cube containing an equal quantity of matter, of the same density? *Ans.* 6426.4 *miles nearly.*

6. Required two mean proportionals between 2 and 54[*]? *Ans.* 6 *and* 18.

---

[*] To find two mean proportionals, divide the greater extreme by the less, and extract the cube root of the quotient; then multiply the less extreme by

7. Required the two mean proportionals between 3 and 375? *Ans.* 15 *and* 75.

8. The earth revolves round the sun in $365\frac{1}{4}$ days, at 95 millions of miles distance from him; required the distance of Jupiter from the sun, supposing he revolves about the sun in $4332\frac{1}{2}$ days[d]? *Ans.* 494109000 *miles nearly.*

8. In the temple of Apollo, in the island of Delos, there was a cubical altar, 3 feet each way, made of the horns of animals forcibly bent and entwined together, said to be the work of Apollo in his infancy; required the side of a cube double, and of another, half of the same? *Ans. side of the double,* 3.77976 *feet: side of the half,* 2.3811 *feet.*

283. EXTRACTION OF ROOTS IN GENERAL, *By Approximation*[e].

RULE I. Call the given number whose root is required to be found, *the number.*

II. Find by trials *a power* nearly equal to the number, and call its root, *the assumed root.*

III. Add 1 to the index of the power, and call the result, *the*

---

the said root for the first mean; and this product by the root for the second: thus in the example $\sqrt[3]{\frac{54}{2}} \times 2 =$ the first mean, and this product multiplied by $\sqrt[3]{\frac{54}{2}} =$ the second mean.

[d] The cubes of the distances of any two planets from the sun, are *as* the squares of the times in which they each revolve round him; in this example therefore $\overline{365\frac{1}{4}}|^2 : \overline{4332\frac{1}{2}}|^2 :: \overline{95 \text{ millions}}|^3 :$ the cube of Jupiter's distance, the cube root of which is the answer: in the same manner the distances of all the other planets may be found, their periodic times, with that of the earth, and its distance from the sun, being known.

[e] Approximation, (from the Latin *ad* to, and *proximus* nearest,) is a continual approach, still nearer and nearer, to the quantity sought; by this method the roots of numbers are found, not exactly, but to any assigned degree of nearness, short of absolute exactness: the rule is in substance the same as that first given by Dr. Hutton, in the first volume of his Mathematical Tracts. There are rules by which the roots of *exact* powers may be accurately determined; but for the roots of high powers, the operations require too much time and labour to be of any real use in practice. An universal investigation of the above rule will be given, when we treat of the resolution of the higher equations in Algebra.

sum; subtract 1 from the index, and call the result, *the difference.*

IV. Multiply *the power* by *the sum,* and *the number* by *the difference,* and add both products together for *the first term.*

V. Multiply *the number* by *the sum,* and *the power* by *the difference,* and add both products together for *the second term.*

VI. Make a rule of three stating, thus; say as *the first term*: is to *the second term* : : so is *the assumed root*: to a fourth number, (found by the rule of three,) which will be the root of the given number nearly.

VII. Involve the root found to the given power, and if the power and given number are nearly equal, the work is finished; but if not, the operation must be repeated, thus;

VIII. Let the root found be called *the assumed root,* and its power *the power,* and proceed with these and the given *number, sum,* and *difference,* as before, whence a root will be obtained still nearer the truth. In this manner the operation may be repeated at pleasure, observing always to use the last found root, and its power, for *the assumed root* and *power.*

EXAMPLES.

1. Required the cube root of 520.

Here 520 = *the number.* 3 = *the index.* 3 + 1 = 4 = *the sum.* 3 − 1 = 2 = *the difference. I find by trials that* 8 *is nearly equal to the cube root of* 520; *therefore* 8 = *the assumed root, and* 8^3 = 512 = *the power.*

Then 512 × 4 = 2048 = *the power multiplied by the sum.*
520 × 2 = 1040 = *the number multiplied by the difference.*
their sum = 3088 = *the first term.*

And 520 × 4 = 2080 = *the number multiplied by the sum.*
512 × 2 = 1024 = *the power multiplied by the difference.*
their sum 3104 = *the second term.*

Then 3088 : 3104 : : 8

```
        8
3088)24832(8.0414 = the root nearly.
     24704
     -----
       12800
       12352
       -----
         4480
         3088
         -----
         13920
         12352
         -----
          1568
```

Proof.

$\overline{8.0414}|^3 = 519.99$, *&c. which is very nearly equal to the given number.*

2. Required the 5th root of 40.

Here 40 = *the number.* 5 = *the index.* 5 + 1 = 6 = *the sum.* 5 − 1 = 4 = *the difference. Let the root found by trials be* 2 = *the assumed root ; then* $\overline{2}|^5 = 32$ = *the power.*

Then 32 × 6 = 192 *and* 40 × 6 = 240
40 × 4 = 160 32 × 4 = 128
the first term = 352 *the second term* = 368

Then $352 : 368 :: 2 : \dfrac{368 \times 2}{352} = 2.0909$, &c. = *the root nearly.*

To repeat the operation.

Here 2.0909 = *the root assumed.* $\overline{2.0909}|^5 = 39.963757$, &c. = *the power.*

Then 6 × 39.963757 = 239.782543
4 × 40 = 160.
First term 399.782543.

6 × 40 = 240.
4 × 39.963757 = 159.855029
Second term 399.855029

Wherefore 399.782543 : 399.855029 : : 2.0909 :

$\dfrac{399.855029 \times 2.0909}{399.782543} = 2.09127910854$3 = *the root extremely near.*

3. What is the cube root of 17.54? *Root* 2.5982, &c.
4. What is the 4th root of 94.75853? *Root* 3.1196, &c.
5. What is the 5th root of 3124? *Root* 4.9996, &c.
6. Extract the 6th root of 48. *Root* 1.90636, &c.
7. Required the 7th root of 581. *Root* 2.4824, &c.
8. Required the 8th root of 72138957.88. *Root* 9.5999, &c.

9. What is the 9th root of 2? *Root* 1.080059, &c.

10. Extract the second, third, fourth, fifth, sixth, seventh, eighth, and ninth roots of one hundred, and find their sum. *Ans.* 27.84716, &c.

284. In all the foregoing examples the index of the root is a fraction, having 1 for its numerator; examples however sometimes occur, in which the numerator of the index is greater than 1; in this case the root is extracted by the following:

RULE I. Involve the given number to that power which is denoted by the numerator of the index, from whence extract the root denoted by the denominator: or,

II. First extract the root denoted by the denominator, then involve this root to the power denoted by the numerator [f].

11. Find the value of $\overline{8}|^{\frac{2}{3}}$.

Thus by Rule I. $\overline{8}|^2 = 64$, and $\sqrt[3]{64} = 4$, *the root required. By Rule* II. $\sqrt[3]{8} = 2$, and $\overline{2}|^2 = 4$, *the root as before.*

12. Required the value of $\overline{10}|^{\frac{4}{5}}$.

Thus $\overline{10}|^4 = 10000$ *and* $\sqrt[5]{10000} = 6.3096$, &c. *the root: or* $\sqrt[5]{10} = 1.5849$, &c. *and* $\overline{1.5849}|^4 = 6.3096$, *the root as before.*

13. Required the value of $\overline{2}|^{\frac{3}{4}}$. *Ans.* 1.68179, &c.

14. Required the value of $\overline{1023}|^{\frac{2}{5}}$. *Ans.* 15.993, &c.

15. Required the value of $\overline{10648}|^{\frac{2}{3}}$. *Ans.* 484.

16. Required the sum of the values of $\overline{2}|^{\frac{2}{3}}$, $\overline{3}|^{\frac{3}{4}}$, $\overline{4}|^{\frac{4}{5}}$, and $\overline{5}|^{\frac{5}{6}}$. *Ans.* 10.72196, &c.

PROGRESSION.

285. When several numbers or terms are placed in regular succession, the whole is called a series.

286. If the terms of a series successively increase or decrease, according to some given law, the series is said to be in progression.

287. Progression is of two kinds, Arithmetical and Geometri-

---

[f] In this rule both Involution and Evolution are employed; the numerator of the fractional index denoting a *power*, and the denominator a *root*; thus in ex. 11. $8^{\frac{2}{3}}$ denotes either *the cube root of the square of* 8, or *the square of the cube root of* 8; on this principle the rule depends.

cal, arising from the manner in which the successive increase or decrease is made; namely, either by addition or subtraction, or by multiplication or division.

ARITHMETICAL PROGRESSION.

288. A series of numbers is said to be in Arithmetical Progression, when the terms successively increase or decrease by the constant addition or subtraction of a number, called the common difference [s].

There are five particulars belonging to questions in arithmetical progression; viz.

1. The least term, } called *the extremes.*
2. The greatest term, }
3. The number of terms.
4. The common difference.
5. The sum of all the terms.

Any three of these five being given, the remaining two may be found, as is shewn by the rules and examples following.

289. *The least term, the greatest term, and the number of terms, being given, to find the sum of all the terms.*

RULE. Add the least and greatest terms together, multiply the sum by half the number of terms, and the product will be the sum required.

EXAMPLES.

1. The least term is 3, the greatest 17, and the number of terms 8, in an arithmetical progression; required the sum of the terms.

Thus $3 + 17 = 20 =$ *sum of the extremes.*
And 4 (*or half* 8) $=$ *half the number of terms.*
Then $20 \times 4 = 80 =$ *the sum required.*

---

[s] When the progression consists of three or four terms *only*, it is usually called an *arithmetical proportion*; and the middle terms are called *arithmetical means*.

The essential property of an arithmetical progression is this; namely, "The sum of the two extreme terms is equal to the sum of every two mean "terms equally distant from the extremes;" from this property many others, some of which are the subject of the following rules, are easily deduced; but as this cannot be conveniently done without Algebra, it was thought best to refer to the Algebraic part of the work for proof of the rules here given. The word *progression* is derived from the Latin *progredior*, to go forward.

2. The least term is 5, the greatest 205, and the number of terms 11, being given, to find the sum of the terms.

Thus $5 + 205 = 210 =$ *sum of the extremes.*
Also $5\frac{1}{2} =$ *half the number of terms.*
Wherefore $210 \times 5\frac{1}{2} = 1155 =$ *the sum required.*

3. The extremes are 4 and 800, and the number of terms 40, to find the sum.

Thus $\overline{4 + 800} \times 20 = 16080$, *the sum required.*

4. A man paid a debt which he owed at 20 payments in arithmetical progression; the first payment was 3*l.* and the last 18*l.* what was the debt? *Ans.* 210*l.*

5. I bought 100 peaches, and paid for them in arithmetical progression, viz. for the first $\frac{1}{4}$*d.* and for the last 6*d.* what sum did the whole amount to? *Ans.* 1*l.* 7*s.* 1*d.*

6. What must be given for 120 elm trees, the prices whereof are in arithmetical progression, that of the first being 5*s.* and that of the last 10*l.* *Ans.* 615*l.*

290. *The least term, the greatest term, and the number of terms, being given, to find the common difference.*

RULE. Subtract the least term from the greatest, and divide the remainder by 1 less than the number of terms; the quotient will be the common difference required.

7. In an arithmetical progression, the least term is 3, the greatest 17, and the number of terms 8; required the common difference?

Thus $17-3 = 14 =$ *the difference of the extremes.*
And $8-1 = 7 =$ *the number lessened by* 1.
Therefore $\frac{14}{7} = 2 =$ *the common difference sought.*

8. The least term is 5, the greatest 205, and the number of terms 11, in an arithmetical progression; required the common difference?

Thus $\frac{205-5}{11-1} = \frac{200}{10} = 20 =$ *the common difference required.*

9. A man had 5 sons, whose ages were in arithmetical progression, the youngest was 3 years old, when the eldest was 13; required the common difference of their ages?

Thus $\frac{13-3}{5-1} = \frac{10}{4} = 2\frac{1}{2}$ *years, the com. difference required.*

10. Bought 120 sheep, and gave a shilling for the first, and five pounds for the last; if the prices are in arithmetical progression, what is the common difference? *Ans.* $9\frac{117}{119}d.$

11. Required the common difference of 40 terms in arithmetical progression, whereof the least is 4, and the greatest 800? *Ans.* $20\frac{16}{39}$.

12. A farmer bought 100 oxen, for the first he paid $3l.$ and for the last $48l.$ supposing the prices in arithmetical progression, what was the common difference? *Ans.* $9s.\ 1\frac{1}{11}d.$

291. *The least term, the greatest term, and the common difference being given, to find the number of terms.*

RULE. Subtract the less term from the greater, and divide the remainder by the common difference; increase the quotient by 1, and it will be the number of terms required.

13. The least term of an arithmetical progression is 4, the greatest 39, and the common difference 5; required the number of terms?

Thus $\frac{39-4}{5} = \frac{35}{5} = 7$. *Then* $7 + 1 = 8$, *the number required.*

14. A grazier sold a certain number of oxen, the prices of which were in arithmetical progression; for the first he received $2l.$ and for the last $50l.$ how many were there, supposing the common difference of the prices to be $4l.$?

Thus $\frac{50-2}{4} + 1 = \frac{48}{4} + 1 = 12 + 1 = 13 =$ *the number required.*

15. The ages of a family are in arithmetical progression, the youngest is 5 years old, the eldest 27, and the common difference 2; required the number of persons?

Thus $\frac{27-5}{2} + 1 = \frac{22}{2} + 1 = 11 + 1 = 12 =$ *the answer.*

16. The two extremes of an arithmetical progression are 27 and 38, and the common difference 1; required the number of terms? *Ans.* 12.

17. A bill was paid by instalments in arithmetical progression, the least payment was 5*l.* the greatest 29*l.* and the common difference 4*l.*; how many payments were made? *Ans.* 7.

18. Bought a lot of books, and paid 1*s.* 6*d.* for the first, 5*s.* 8*d.* for the last, and each (beginning at the first) cost 5*d.* more than the preceding; how many books were there? *Answer,* 11.

292. *Given the number of terms, the sum of the terms, and the common difference, to find the least term.*

RULE I. Divide the sum of the terms by the number of terms.

II. Subtract 1 from the number of terms, and multiply the remainder by half the common difference.

III. From the quotient (found above) subtract this product, and the remainder will be the least term.

19. The sum of an arithmetical progression, consisting of 11 terms, is 154, and the common difference 2; required the least term?

Thus $\frac{154}{11} = 14$ *the quotient of the sum, by the number of terms.*

And $\overline{11-1} \times \frac{2}{2} = 10 =$ *the number of terms minus* 1, *multiplied by half the common difference.*

Then $14 - 10 = 4$, *the least term, as was required.*

20. The sum of the terms 366, the number of terms 12, and the common difference 5, of an arithmetical progression being given, required the least term?

Thus $\frac{366}{12} = \frac{61}{2}$. *And* $\overline{12-1} \times \frac{5}{2} = 11 \times \frac{5}{2} = \frac{55}{2}$.

Therefore $\frac{61}{2} - \frac{55}{2} = \frac{6}{2} = 3 =$ *the least term.*

21. The sum of the ages of 9 persons is 162, and the common difference 3 years; required the age of the youngest?

Thus $\frac{162}{9} - \overline{9-1} \times \frac{3}{2} = 18 - 12 = 6$, *the answer.*

22. The sum of 6 numbers in arithmetical progression is 108,

and the common difference 4; required the least term? *Answer*, 8.

23. Seven poor persons received among them 63 shillings, their shares were in arithmetical progression, the common difference being 2; required the least share? *Ans.* 3 *shillings.*

293. *Given the least term, the number of terms, and the common difference, to find the greatest term.*

RULE. Multiply the number of terms by the common difference, to the product add the least term, and from this sum subtract the common difference; the remainder will be the greatest term.

24. In an arithmetical series of 10 terms, the least term is 8, and the common difference 3; required the greatest term?

Thus $10 \times 3 + 8 - 3 = 30 + 8 - 3 = 38 - 3 = 35 =$ *the greatest term.*

25. The least term of an arithmetical series is $2\frac{1}{2}$; the common difference $4\frac{1}{2}$; and the number of terms 16: required the greatest term?

Thus $16 \times 4\frac{1}{2} + 2\frac{1}{2} - 4\frac{1}{2} = 70 =$ *the greatest term.*

26. A man has 12 children, the youngest is three quarters of a year old, and each was born when the preceding was fifteen months old; required the age of the eldest?

Thus $12 \times 1\frac{1}{4} + \frac{3}{4} - 1\frac{1}{4} = 14\frac{1}{2}$, *the answer.*

27. The least term is 3, the common difference 2, and the number of terms 7; to find the greatest term? *Ans.* 15.

28. The age of the youngest of 9 persons is 6, and the common difference 3; required the age of the eldest? *Ans.* 30.

29. The least term is $2\frac{1}{4}$, the common difference $3\frac{1}{4}$, and the number of terms 20, being given to find the greatest term? *Ans.* 64.

294. PROMISCUOUS EXAMPLES FOR PRACTICE.

1. How many times does the hammer of a clock strike in 12 hours? *Ans.* 78.

2. The clocks at Venice go to 24 o'clock; how many times does the hammer of one of them strike the bell in that space of time? *Ans.* 300.

3. If 1000 stones be placed in a straight line, the first a yard

distant from a basket, and the rest in succession, each a yard distant from the preceding; what length of ground must a man go over, to pick up the stones one by one, and return with them singly to the basket? *Ans.* 560 *miles*, 1540 *yds.*

4. Bought 9 books, the prices in arithmetical progression, that of the least being 3 shillings, and that of the best 19; what sum did I pay for the whole; and what is the common difference of the prices? *Ans. Paid* 4*l.* 19*s.* *Com. diff.* 2*s.*

5. A man travelled 2 miles the first day, and 53 miles the last, and increased every day's journey three miles more than the preceding; how many days, and what distance, did he travel? *Ans.* 18 *days, and travelled* 495 *miles.*

6. There are 64 squares on a chess board; now if I lay half a crown on the first square, three shillings on the second, and so on, increasing successively by sixpence, how much will there be on the last square, and on the whole board? *Ans. On the last square* 1*l.* 14*s.* *On the whole board* 58*l.* 8*s.*

7. A debt of 15*l.* is to be discharged at 12 payments, each succeeding payment to exceed the former by 4 shillings; what will the first and last payment be? *Ans. The first payment* 3 *shillings.* *The last* 2*l.* 7*s.*

8. A poet, who had agreed with a bookseller to receive 40*l.* for every thousand verses he should write, set to work on new year's day, and composed 10 verses; next day he composed 12, and so on, increasing every day by 2; now allowing 70 days for sundays and other holidays, what sum would be due to him at the year's end?

GEOMETRICAL PROGRESSION.

295. A series of numbers is said to be in Geometrical Progression, when the terms successively increase by the constant multiplication by some number called *the ratio*, or decrease by constantly dividing by the same number or *ratio* [h].

---

[h] The fundamental property of a Geometrical Progression is this; namely, "The product of the two extreme terms is equal to the product of any two intermediate terms, equally distant from the extremes;" from this property the rest are derived, as will be shewn when we resume the subject in the Alge-

There are five particulars which belong to questions in geometrical progression; namely,

1. The least term, } called *the extremes*.
2. The greatest term, }
3. The number of terms.
4. The ratio.
5. The sum of all the terms.

Any three of these five being given, the remaining two may be found, as follows.

296. The least term, the greatest term, and the ratio being given, to find the sum of the series.

RULE 1. Multiply the greatest term by the ratio, and from the product subtract the least term for a dividend.

2. Subtract 1 from the ratio for a divisor.

3. Divide the dividend by the divisor, and the quotient will be the sum required.

EXAMPLES.

1. The least term is 2, the greatest 6250, and the ratio 5, in a series in geometrical progression; required the sum of the series?

Thus $6250 \times 5 - 2 = 31250 - 2 = 31248$ *the dividend.*

And $5 - 1 = 4$ *the divisor.*

Therefore $\frac{31248}{4} = 7812$ *the sum required.*

2. The least term 1024, the greatest 59049, and the ratio $1\frac{1}{2}$, being given in a geometrical series, to find the sum of all the terms?

Thus $59049 \times 1\frac{1}{2} - 1024 = 87549.5$ *dividend.*

And $1\frac{1}{2} - 1 = \frac{1}{2} = .5$ *divisor.*

Then $\frac{87549.5}{.5} = 175099$ *the sum required.*

3. Given in a geometrical progression the least term 10, the greatest term 10000, and the ratio 6, required the sum?

---

braic part of the work; the grounds of the several rules in arithmetical and geometrical progression, cannot be explained in a satisfactory manner by common arithmetic.

Thus $\frac{10000 \times 6 - 10}{6 - 1} = \frac{59990}{5} = 11998$ *the sum.*

4. The least term is 1, the greatest 2187, and the ratio 3, required the sum of the series? *Ans.* 3280.

5. The extremes are 10 and 100, and the ratio $1\frac{1}{4}$, in a geometrical progression; required the sum? *Ans.* 550.

6. The extremes of a geometrical progression are 5 and 320, and the ratio 2; required the sum? *Ans.* 635.

297. *Given the greatest term, number of terms, and ratio, to find the least term.*

RULE. Involve the ratio to the power whose index is 1 less than the number of terms, divide the greatest term by this power, and the quotient will be the least term.

7. In a geometrical series, the greatest term is 972, the number of terms 6, and the ratio 3, required the least term?

Here $6 - 1 = 5$. *then* $\overline{3}|^5 = 243$. *wherefore* $\frac{972}{243} = 4$, *the least term required.*

8. Required the least term of a geometrical progression, of which the greatest term is 1536, the ratio 2, and the number of terms 10?

Thus $10 - 1 = 9$. *then* $\overline{2}|^9 = 512$. *wherefore* $\frac{1536}{512} = 3$, *the least term.*

9. In a geometrical progression the greatest term is 10.2487, the ratio 1.1, and the number of terms 5; required the least term?

Thus $5 - 1 = 4$, *and* $\overline{1.1}|^4 = 1.4641$, *and* $\frac{10.2487}{1.4641} = 7$, *the least term.*

10. In a geometrical progression consisting of 6 terms, the greatest term is 1024, and the ratio 4; required the least term? *Ans.* 1.

11. Given the greatest term 768, number of terms 9, and ratio 2, to find the least term? *Ans.* 3.

298. *The least term, ratio, and number of terms being given, to find the greatest term.*

RULE. Involve the ratio to that power whose index is one

less than the number of terms; multiply the power by the least term, and the product will be the greatest term.

12. The least term of a geometrical progression is 3, the ratio 2, and the number of terms 9; to find the greatest term?

Thus $9 - 1 = 8$, *then* $\overline{2}|^8 = 256$, *whence* $256 \times 3 = 768$, *the greatest term.*

13. In a geometrical series of 5 terms, the least term is 10, and the ratio 7, required the greatest term?

Thus $5 - 1 = 4$, *and* $\overline{7}|^4 = 2401$, *whence* $2401 \times 10 = 24010$, *the greatest term.*

14. The least term of a geometrical series is 8, the ratio 3, and the number of terms 7; to find the greatest term?

Thus $\overline{3}|^6 \times 8 = 729 \times 8 = 5832$, *the greatest term.*

15. In a geometrical progression there are given the least term 2, the ratio 3, and the number of terms 4; to find the greatest term? *Ans.* 54.

16. Required the greatest term of a geometrical series, whose least term is 5, ratio 6, and number of terms 7? *Ans.* 233280.

299. *The two extremes, and the sum of the series being given, to find the ratio.*

RULE. Subtract the least term from the sum, and also the greatest from the sum; then divide the former remainder by the latter, and the quotient will be the ratio required.

17. In a geometrical progression, the extremes are 10 and 10000, and the sum is 11998, required the ratio?

Thus $11998 - 10 = 11988$. *and* $11998 - 10000 = 1998$.

Then $\frac{11988}{1998} = 6$, *the ratio required.*

18. The extremes are 1024 and 59049, and the sum 175099; required the ratio?

Thus $175099 - 1024 = 174075$, *and* $175099 - 59049 = 116050$, *whence* $\frac{174075}{116050} = 1\frac{1}{2}$, *the ratio required.*

19. To find the ratio of a geometrical progression, whose sum is 550, and the extremes 10 and 100?

Thus $550 - 10 = 540$, *and* $550 - 100 = 450$, *whence* $\frac{540}{450} = 1\frac{1}{5}$, *the ratio required.*

20. The extremes are 1 and 2187, and the sum of the series 3280; required the ratio? *Ans.* 3.

21. The sum of a geometrical series is 635, and the extremes are 5 and 320, required the ratio? *Ans.* 2.

300. *The least term and ratio being given, to find any proposed term of the series.*

RULE I. Write down a few of the leading terms of the given geometrical series, and place over them as indices the terms of an increasing arithmetical series, whose common difference is 1, namely 1, 2, 3, 4, 5, &c. when the least term and ratio of the given series are equal; and 0, 1, 2, 3, 4, 5, &c. when they are unequal.

II. Add together such of the indices as will make the index of the term required; if the least term and ratio are equal, this index will be equal to the number denoting the place of that term; but if they are unequal, the index will be 1 less.

III. Multiply those terms of the geometrical series together, which stand under the indices added, and the product will be the term sought, when the first (or least) term and ratio are equal [1].

IV. But if the first term be not equal to the ratio, involve the first term to the power whose index is 1 less than the number of terms multiplied, divide the above product by this power, and the quotient will be the term required.

22. The first term of a geometrical series is 2, the ratio 2, and the number of terms 14; required the last or greatest term?

OPERATION.

Thus	1.	2.	3.	4.	5	*indices.*
And	2.	4.	8.	16.	32	*leading terms.*

Then $2 + 3 + 4 + 2 + 3 = 14 =$ *index of the* 14*th term.*

[1] This property of the indices is the foundation of Logarithms; its use in this place is extremely obvious: for knowing the last term, we also know what its index will be; and knowing the index, we readily perceive what terms of the *arithmetical* series must be added together to produce it, and these terms indicate what terms in the *geometrical* series are to be multiplied together to produce the last term.

To try to account for this mutual correspondence of the two progressions, would be a vain attempt; like many other properties, it follows from the nature of numbers, and this is perhaps all that can be said on the subject.

And $4 \times 8 \times 16 \times 4 \times 8 = 16384 =$ *the* 14*th term, or answer.*

Explanation.

The first term and ratio being equal, I take the series 1. 2. 3. 4, &c. (beginning with 1) for indices: under these I place the leading terms 2. 4. 8, &c. of the given geometrical series; then, because the index of the term required is evidently 14, I choose any of the indices, which added together make 14, namely, 2. 3. 4. 2 and 3; I then multiply the terms which stand under these together, namely, 4. 8. 16. 4 and 8, and the product is the answer required.

23. Required the 20th term of the series 1. 3. 9. 27, &c.?

OPERATION.

Thus 0. 1. 2. 3. 4. 5 *indices.*

And 1. 3. 9. 27. 81. 243 *leading terms.*

Then $5 + 5 + 4 + 3 + 2 = 19$ *index of the* 20*th term.*

And $243 \times 243 \times 81 \times 27 \times 9 = 1162358667$ *the* 20*th term.*

Explanation.

The first term and ratio not being equal, the indices must begin with 0, and consequently 19 will be the index of the 20th term. But by the rule, the first term ought to have been involved to the 4th power, (one less than 5, the number of terms multiplied,) by which the product of the terms should have been divided; this is omitted, because the first term being 1, all its powers will be 1, and dividing by 1 makes no alteration.

24. What is the 10th term of the series 5. 10. 20. 40, &c.?

OPERATION.

Thus 0. 1. 2. 3. 4 *indices.*

And 5. 10. 20. 40. 80 *leading terms.*

Then $4 + 3 + 2 = 9$ *index of the* 10*th term.*

Whence $80 \times 40 \times 20 = 64000$ *dividend.*

Also $\overline{5}|^2 = 25$ *the divisor. wherefore* $\frac{64000}{25} = 2560$ *the* 10*th term required.*

Explanation.

The first term 5, and ratio 2, being unequal, I divide the product of the terms, viz. 64000 by $\overline{5}|^2$ or 25: that is, by that power of the first term 5, whose index 2, is less by 1 than the number of terms 3, multiplied together.

25. What is the 11th term of the series 1. 2. 4. 8. 16, &c.? *Ans.* 1024.

26. Required the 13th term of the series 2. 4. 8. 16. 32, &c.? *Ans.* 8192.

27. The first term of a geometrical progression is 5, and the ratio 3; required the 13th term? *Ans.* 2657205.

301. PROMISCUOUS EXAMPLES FOR PRACTICE.

1. Nine sea officers divide a prize, the first receives 20*l.* the second 60*l.* and so on in triple proportion; what sum will the Admiral (who has the largest share) receive? *Ans.* 131220*l.*

2. Bought 12 pigs, and paid a farthing for the first, a half-penny for the second, and so on, doubling continually the price of the last; what did they cost me? *Ans.* 4*l.* 5*s.* 3*d.*$\frac{3}{4}$.

3. A servant agreed with his master for 12 months, to receive a farthing for the first months' service, a penny for the second, 4*d.* for the third, &c. what sum did his wages amount to? *Answer,* 5825*l.* 8*s.* 5*d.*$\frac{1}{4}$.

4. The profits of a certain trading company, which has been established 12 years, have increased yearly in geometrical progression; the gain of the first year was 5*l.* and that of the year just expired 885735*l.* required the ratio of increase, and the sum of the profits? *Ans. the ratio* 3. *the sum* 1328600*l.*

5. A person of property in Ireland agreed with Government to exert his influence, to procure seamen for the navy; the first month he sent over 1 man, the second 2 men, the third 4, and so on in geometrical progression; what number did he send over in 15 months, and how many in the last month of that time? *Ans. sent in all* 32767 *men: in the last month* 16384.

6. Suppose a laceman agrees to sell 22 yards of lace at the rate of 2 pins for the first yard, 6 for the second, and so on in triple proportion; what sum will he receive for the whole, allowing the pins to be worth a farthing a hundred? *Answer,* 326886*l.* 0*s.* 9*d.*

7. What sum would a horse sell for that has 4 shoes on, with 8 nails in each shoe, at 1 farthing for the first nail, 2 for the second, 4 for the third, and so on? And what would be the price of another horse, having only two shoes, on the same conditions? *Ans.* 4473924*l.* 5*s.* 3*d.*$\frac{3}{4}$ *the first: and* 68*l.* 5*s.* 3*d.*$\frac{3}{4}$ *the last.*

PART II.

LOGARITHMS.

HISTORICAL INTRODUCTION.

1. [a] LOGARITHMS are a series of numbers in arithmetical progression, adapted to another series in geometrical progression, in such sort that 0 in the former series always corresponds to 1 in the latter, and the succeeding terms of the former to the succeeding terms of the latter, each to each.

2. The use of Logarithms is to lessen the labour and time which long calculations performed by common numbers necessarily require, addition and subtraction by Logarithms performing multiplication and division by numbers, &c. so that an operation may be performed in a few minutes by Logarithms, which would sometimes require as many hours by common arithmetic.

3. But the advantages attending the use of Logarithms would be very limited, if these useful numbers were exclusively confined to a geometrical progression; the common numbers not being in geometrical, but in arithmetical progression: this defect has been happily supplied by an admirable contrivance, which will be explained in its proper place, whereby Logarithms are extended to the entire algorithm of numbers, every number, whether integral or fractional, having its proper Logarithm.

[a] The word *Logarithm* is derived from the Greek λογος, *ratio*, and αριθμος, *number*, and implies either the *ratio of numbers*, or *number of ratios*, both interpretations being descriptive of the nature of Logarithms.

4. The fundamental property of Logarithms is this; if an arithmetical progression be applied to a geometrical one, in the manner above stated, the terms of the former will be indices to those of the latter: now if any two of these indices be added together, the sum will be the index of the product of the two numbers corresponding to those indices: if one index be subtracted from another, the remainder will be the index of the quotient which arises by dividing the number corresponding with the former, by that corresponding with the latter: if an index be multiplied by any number, the product will be the index of the term which is the power denoted by that number; and if an index be divided by any number, the quotient will be the index of the root denoted by that number. This property of the two progressions was known to the ancients, and treated of by Euclid and Archimedes. Stifelius in his *Arithmetica Integra*, printed at Nuremberg in 1544, explains it at large, shewing its use and application in a great variety of instances; so that it seems this author was in possession of the general idea of Logarithms, although under another name: and the reason assigned for his not computing Tables is, that he was not under a necessity of performing those long and troublesome calculations, which require the aid of Logarithms. Justus Byrgius[b], and Longomontanus[c], are re-

[b] Justus Byrgius was a French Mathematical Instrument Maker, and assistant Astronomer to the Landgrave of Hesse; the invention of the Sector is ascribed to him, as was that of Logarithms, but the latter has never been proved: he flourished in the latter part of the 16th century.

[c] Christian Longomontanus was born at a village of the same name in Denmark, in 1562; his father's name was *Severin*, and it is remarkable, that notwithstanding the obscurity of his father and his birth-place, he has contrived to dignify and eternize them both, by stiling himself in the title-page of some of his works, *Christianus Longomontanus, Severini Filius*. His parents being very poor, obliged him to work for his daily support, but he occasionally took lessons of the parish priest; at length he eloped, and went to the College at Wyburg, where he spent eleven years, being obliged to work for his living, and study

ported to have known and constructed these numbers; but the person to whom the world is indebted for the first publication of them was, John Lord Napier[e], Baron of Merchiston, in Scotland, in a work intitled, *Mirifici Logarithmorum Canonis Descriptio*, printed in 1614. This work contains Tables of the Logarithms of Numbers, and of the Logarithmic Sines, Tangents, and Secants, for every minute of the quadrant, with definitions, description of the Tables, &c. but the Author chose to omit giving the method of construction, until the opinion of the learned concerning his invention should be ascertained. This discovery immediately excited the attention of mathematicians, and a translation of Napier's Book into English was made by Mr. Edward Wright[d], the inge-

alternately: he afterwards spent eight years as a very useful assistant to the celebrated Astronomer, Tycho Brahe; and at length obtained the Professorship of Mathematics at Copenhagen, where he died in 1647.

His principal work is entitled, *Astronomica Danica*, 4to. 1621. and fol. 1640.

[e] John Lord Napier, (or Neper, as he is sometimes called,) was born in 1550, and was educated at the University of St. Andrews; he made the tour of Europe, and after his return applied himself closely to literature and science. Mathematics, especially astronomy, appear to have been his favorite study; and the numerous and intricate calculations requisite in the latter branch, put him upon various contrivances for shortening the work, which proved the source of the noble invention of Logarithms. He was the inventor of the instrument called *Napier's Bones*, consisting of five rulers of bone, wood, pasteboard, or ivory, whereby the arithmetical operations of multiplication, division, &c. may be performed *mechanically* with great ease; a full description of which he published in 1617, in a work entitled *Rabdologia, seu Numerationis per Virgulas libri duo*. Napier likewise invented the rule for the five Circular Parts in spherical Trigonometry; he died at Merchiston, in 1617.

[d] Edward Wright, Esq. lived at the end of the 16th, and beginning of the 17th century. He was deeply skilled in mathematics and mechanics, and is justly celebrated as the inventor of what is erroneously called Mercator's Chart, having first discovered the true method of dividing the meridian line, on which Mercator's projection is founded; the principles of which he clearly shewed in *The Correction of certain Errors in Navigation*; a work, which, although written many years before, was not published till 1699. Every thing valuable in the celebrated maps of Jodicus Hondius was derived from the instructions given him on the subject by Wright; nevertheless the former with unbecom-

nious inventor of the principles of Mercator's Sailing, which was published in 1616, after his death, by his Son, Samuel Wright, with a dedication to the East India Company, and a preface by Mr. Henry Briggs*, at that time Professor of Geometry at Gresham College. In 1619, two years after Lord Napier's death, a new edition of his work was published by his Son, Robert Napier, containing the construction of his Canon, and other miscellaneous pieces omitted in the first edition. On the first publication of Lord Napier's invention, Mr. Briggs paid him a visit, and the result of their communication on the subject, was a determination to change the form of the Logarithms for one better adapted to the Decimal scale of Numbers: this alteration we have reason to believe was first suggested by Briggs, as he was the first who published Logarithms on the improved plan, and, it seems, entertained a hope, that a just acknowledgment would be publicly made of the part he had taken in the improvement; in this he was disappointed,

ing ingratitude laid claim to the invention. Our author published a work on the Sphere, another on Dialling, and another, very useful to navigators, entitled, *The Haven-finding Art:* he was the inventor of several instruments useful in their time, for finding the altitude, &c. of celestial objects, and thence the true place of a ship. He was fellow of Gonvil and Caius College, Cambridge; occasionally read lectures on nautical and mathematical subjects; and was Tutor to Prince Henry. After a life spent in the extension of useful knowledge, he died at London in 1615.

* Henry Briggs was born at Warleywood in Yorkshire, in 1556: at a proper age he was sent to St. John's College, Cambridge, where after taking the degree of M. A. he was chosen a Fellow in 1588; and, in consequence of his great proficiency in mathematical learning, was appointed Examiner and Lecturer in that faculty. In 1596 he was chosen the first Professor of Geometry at Gresham College: in 1619, he was appointed the first Savilian Professor of Geometry at Oxford; in consequence of which, the next year he resigned the Professorship at Gresham College. Besides the works above mentioned, Mr. Briggs was the author of several others equally creditable to his memory. This truly great man terminated a laborious and useful life in January 1631; and was buried in the Choir of the Chapel of Merton College; at which College he had for several years past been a constant resident.

for in the second edition of Napier's work, although the alteration is adverted to, no mention is made of any assistance received from Briggs, either by Lord Napier or his Son. If Briggs was really the inventor of the improvement, as it is generally believed he was, the omission was certainly an act of gross injustice.

5. An improved form of Napier's Logarithms, by Mr. John Speidell, came out the same year; and the year following Justus Byrgius published Tables, in which the natural numbers and Logarithms are arranged in a converse order of what they are in our ordinary Tables. Vincent printed a copy of Napier's work at Lyons, as did Ursinus at Cologne; the latter being improved by the addition of Tables of proportional parts. In 1624, the celebrated Kepler published at Marpurg his *Chilias Logarithmorum, &c.* in which the Logarithms are more conveniently adapted to common numbers than those of Napier; the latter being principally accommodated to the sines of arcs, &c.

6. The Logarithms published by these and some others about the same time, were of the kind which has since been called *hyperbolical;* a name they received in consequence of their expressing the spaces included between the asymptote, and curve of the hyperbola.

7. To have an adequate idea of the nature of Logarithms, we must consider them as indices, denoting the ratios of numbers to unity. Napier's Logarithm of 10, (that is, the index denoting the ratio of 10 to 1,) is 2.3025851, &c. this Mr. Briggs found deficient in point of simplicity and convenience, and therefore the latter gentleman undertook the laborious task of computing an entire new system, in which he made 1 the Logarithm of 10; whence it follows that 2 will be the Logarithm of 100, 3 of 1000, &c. This has been justly consi-

dered as an improvement of the greatest value; it introduces a kind of similarity between the series of Logarithms and that of common numbers, simplifies the whole doctrine amazingly, and renders it plain and intelligible to the meanest capacity.

8. The first fruit of Mr. Briggs's labours in this way, was his *Logarithmorum Chilias Prima*, which appeared in 1617, after Napier's death, containing the first thousand Logarithms to eight places of figures, besides the index. Mr. Edmund Gunter[f] adapted Mr. Briggs's Logarithms, first of any, to the sines and tangents; he computed them for every minute to seven places of figures besides the index; this work appeared in 1620, under the title of *A Canon of Triangles*, which work was reprinted in 1623, with the addition of the Chilias Prima of Briggs. The same year Gunter applied the Logarithms of Numbers, Sines, Tangents, &c. to a straight ruler, whereby computations may be performed by a pair of compasses only: this instrument is still known by the name of *Gunter's Scale*. Other methods of projecting these numbers on circular, sliding, and spiral instruments, were afterwards invented by Wingate, Oughtred, Milburne, and Partridge.

[f] Edmund Gunter was born in 1581, and received the rudiments of his education at Westminster School, under the famous Dr. Busby; from thence he went to Christ Church, Oxford, where in 1615 he took the degree of B. D. in 1619 he succeeded Mr. Williams as Professor of Astronomy at Gresham College, where he greatly distinguished himself by his eminent mathematical talents, displayed in his writings and lectures; he died in 1626. Mr. Gunter's inventions and improvements in mixed mathematics were of the greatest value; in 1606, he gave a new projection of the sector, and in 1618, a new portable quadrant for the more easily finding the hour, azimuth, &c. He discovered in 1622 the changeable declination of the magnetic needle, shewing that it had altered 5 degrees in 42 years; which conclusion was verified by his successor, Mr. Gellibrand. He introduced the scale and measuring chain known by his name, and gave ample descriptions of their uses. He introduced the name *co-sine*, and the use of the *arithmetical complements* of Logarithms; and the first idea of the logarithmic curve is generally ascribed to him.

9. In 1624, Mr. Briggs published his *Arithmetica Logarithmica*, containing the Logarithms of Numbers from 1 to 20000, and from 90000 to 100000, with ample directions for their use, and an earnest invitation to Mathematicians to assist in the completion of the work, by computing the intermediate numbers: this was effected soon after by Adrian Vlacq, of Gouda in Holland, who, besides supplying the intermediate chiliads, added Tables of artificial sines, tangents, and secants, for every minute of the quadrant. This ingenious person printed likewise at Gouda, in 1633, a work entitled *Trigonometria Artificialis*, containing Briggs's Table of the first 20000 Logarithms, with the Logarithmic sines and tangents, and their differences, for every ten seconds of the quadrant, to ten places of figures, with their description and use[g]. At the same time and place was printed Mr. Briggs's *Trigonometria Britannica*, under the superintendence of Vlacq; this work contains the Logarithms of 30000 natural numbers, logarithmic sines, and tangents, for the hundredth part of every degree, all to 14 places of figures besides the index, the natural sines for the same parts to 15 places, and the tangents and secants for the same to 10 places, with the construction of the whole: but the Author dying in 1630, before the work was complete, his friend, Mr. Henry Gellibrand[h], Pro-

[g] This work, which is justly considered as very useful in astronomical calculations, has been lately reprinted at Leipsic, by Vega, under the title of *Thesaurus Mathematicus*.

[h] The Reverend Henry Gellibrand was born in London, in 1597; he was sent to Trinity College, Oxford, in 1615; and in 1623, took the degree of M.A. having taken orders, he became curate of Chiddingstone in Kent; but happening to hear a lecture on the mathematics by Sir Henry Saville, he relinquished all prospect of preferment in the Church, and set himself in earnest to study that noble science. He became Professor of Astronomy at Gresham College, upon the death of Mr. Gantor, in 1627; and was the author of several useful pieces, chiefly tending to the improvement of Navigation, a branch to which his attention was principally directed: he died in 1636.

fessor of Astronomy at Gresham College, supplied the preface, and the application to plain and spherical Trigonometry [i], &c. Two years after, Mr. Gellibrand published *An Institution Trigonometricall*, being a smaller work of the same kind, with the addition of other tables, &c. the whole adapted to the use of navigators.

10. Mr. Bonnycastle, Professor of Mathematics at the Royal Military Academy, Woolwich, has lately discovered an ingenious improvement in the Binomial Theorem of the illustrious Newton, whereby he has shewn the method of constructing Logarithms in a new and elegant manner: several authors, as Gunter, Huygens, Keill, Newton, Mersenne, James Gregory, Mercator, &c. gave methods of computing Logarithms, derived from their analogy to certain curves; others, as Cotes, Halley, Craig, John Bernoulli, Dr. Brook Taylor, Mr. Jones, &c. employed for that purpose either a fluxional process, or methods nearly similar to that of Fluxions; but their methods, although ingenious and scientific, are not strictly conformable to the nature of the subject, which is purely arithmetical. The theorems delivered by Mr. Bonnycastle are unexceptionable in this respect, being derived from the principles of pure Algebra, and by means of them the Logarithms, according to Napier, Briggs, or any other system, are readily obtained.

11. As the Logarithms of Napier have obtained the name of *hyperbolic*, so those of Briggs are usually denominated *common* Logarithms, from the circumstance of their being better adapted to practice than Napier's, and therefore most in use [k]. The following authors, besides

[i] In the *Trigonometria Britannica*, Mr. Briggs has shewn the method of generating the coefficients of the terms of any power of a binomial successively from each other, independent of any other power; which is the foundation of Sir Isaac Newton's celebrated Binomial Theorem.

[k] Besides the common and hyperbolic Logarithms, there are *logistic* and *pro-*

those already mentioned, have treated on the subject; viz. Henrion, Miller, Norwood, Cavallerius, Frobenius, Carumel, Sharp, Leibnitz, Long, Simpson, Wolfius, Maclaurin, Reid, Dodson, Wallis, Maseres, and many others.

It will be proper to observe that Mr. Wingate, as early as 1626, made a very useful improvement in the method of arranging the Logarithms in the Tables, by placing the unit figures of the natural numbers along the tops of the columns, the tens down the margin, and the whole number in the angle of meeting. The Rev. Nathaniel Roe[k] reduced the Tables to a more convenient form in 1633; and about 25 years after, Dr. John Newton[l], availing himself of both these improvements, introduced the method of arrangement which is at present used by the best writers.

12. Of the more modern Tables, Sherwin's and Gardiner's are still in great repute; the former as the most complete collection, the latter as the most correct. Gardiner's Tables, with additions and improvements, were

portional Logarithms. Logistic Logarithms are such as arise from subtracting the common Logarithms from 3.5563 (the logarithm of the number of seconds in 60 minutes). These Logarithms are frequently used for astronomical computations, in any proportion where the first term, or either of the means, happens to be 60 minutes. Proportional Logarithms arise by subtracting the common Logarithms from 4.0334, (or the Logarithm of 10800, the number of seconds in 180 minutes;) these Logarithms, like the former, have their application in astronomy.

[k] Stiled by Dr. Hutton, "Pastor of Benacre in Suffolke."

[l] Dr. Newton was descended from a respectable family in Northamptonshire, where he was born in 1622. He entered a commoner at St. Edmund Hall, Oxford, in 1637: here he acquired a great proficiency in mathematics, and other branches of learning; and after passing through his degrees in Arts, he was created Doctor in Divinity in 1661. Shortly after he was made one of the King's Chaplains, and obtained the rectory of Ross in Herefordshire; this he held till his death, which happened on Christmas day, 1678. His works, which are on Arithmetic, Astronomy, Geography, Logarithms, and other branches, sufficiently evince his skill as a mathematician.

printed at Paris in 1785, by M. Callet, under the title of *Tables Portatives de Logarithmes*, forming a neat octavo volume. The Tables of Mr. M. Taylor, with an excellent introduction by the Rev. Neville Maskelyne, D. D. F. R. S. Astronomer Royal, printed in 1792, are deservedly esteemed[m]; but the most useful collection of any are Dr. Hutton's *Mathematical Tables*, published in 1785; this work contains the Logarithms of numbers from 1 to 100000, to 7 decimals; Logarithms to 20 places; Logarithms to 61 places; an antilogarithmic table to 20 places[n]; hyperbolic and logistic Logarithms; natural and artificial sines, tangents, secants, and versed sines; traverse table; points of the compass; arcs, &c. &c. preceded by an elaborate introduction, shewing the construction and uses of the tables, together with an historical account of the invention and improvements of Logarithms. From this work some of the particulars given above were taken, and to it the reader is referred for many interesting particulars, which could not be introduced in this place.

On the method of applying Logarithms to the common, or natural numbers.

13. We have defined Logarithms as a series of numbers in arithmetical progression, adapted to another series in geometrical progression, so that 0 in the former being placed over 1 in the latter, 1 in the former over 10 in the latter, &c. the terms of the former series will

[m] The Tables of Mr. M. Taylor are the most extensive of any that have hitherto appeared, and therefore the best adapted for subjects where extreme accuracy is required.

[n] The first instance of an antilogarithmic canon, was that begun, as it is believed, by Thomas Harriot, who died in 1621, and finished by Warner, about 1640; but the work was never printed. Mr. Dodson's antilogarithmetic canon was published in London in 1742, and contains the numbers corresponding to every logarithm, from 1 to 100000, to eleven places, with their differences and proportional parts.

be the indices or *logarithms* of the respective terms of the latter; thus,

0. 1. 2. 3. 4. *Arithm. series, or logarithms.*

1. 10. 100. 1000. 10000. *Geom. series, or natural numb.*

that is, 0 is the log. of 1. . 1 the log. of 10. . 2 the log. of 100, &c. Because 0 is the log. of 1, and 1 the log. of 10, it follows that any numbers between 1 and 10 will have 0 with some decimal for its logarithm; in like manner any number between 10 and 100 will have 1 and some decimal for its logarithm; any number between 100 and 1000 will have 2 and some decimal for its logarithm, &c.

Thus the log. of 7, (which is between 1 and 10,) is 0.8450980; the log. of 75, (a number between 10 and 100,) is 1.8750613; the log. of 354, (between 100 and 1000,) is 2.5490033, &c.

14. Every logarithm, then, consists of a whole number and a decimal, or has their places supplied by one or more ciphers.

15. The whole number is called the *index*, or *characteristic* of the logarithm; it points out the value of the left hand figure of the number corresponding to the logarithm, by shewing its distance from unity, and will be *affirmative* or *negative*, according as the correspondent number is integral or fractional.

Thus, if the two foregoing series be inverted, and continued backward to any length, we shall have

3. 2. 1. 0. −1. −2. −3, &c. *the logarithms.*

1000. 100. 10. 1. .1. .01. .001, &c. *the numbers.*

where 3 is the log. of 1000. . . 2 of 100. . . 1 of 10 . . 0 of 1. . . −1 (*minus* 1) of .1 or $\frac{1}{10}$. . . −2 (*minus* 2) of .01 or $\frac{1}{100}$. . . −3 (*minus* 3) of .001 or $\frac{1}{1000}$, &c.

Hence if a whole number contain 4 places of figures, the index of its logarithm will be 3—. . if it contain 3 places, the index will be 2; if 2 places, the index will be 1; and if 1 place, the index will be 0. And in decimals, if

the first significant figure be *tenths*, the index will be -1, if *hundredths* -2, if *thousandths* -3, &c. and when a whole number is connected with a decimal, the index of the *highest place of the whole number* will be the proper index: so that *in every case* the index shews the value of the left hand significant figure, by pointing out its distance from unity; if the index be *affirmative*, it characterizes a whole or mixed number; if *negative*, a decimal.

16. As the characteristic, or index, shews the limits between which its correspondent number is, namely, whether it be among the *units*, the *tens*, the *hundreds*, &c. or among the *tenths*, the *hundredths*, the *thousandths*, &c. so the decimal part of the logarithm marks the exact point within the determined limits, to which the said number belongs; consequently, both together will determine that number *exactly*. For example; the number 2 occupies the same place between 1 and 10, that 20 does between 10 and 100, and that 200 does between 100 and 1000, and that 2000 does between 1000 and 10000, &c. therefore the decimal part of the logarithm of 2, of 20, of 200, of 2000, &c. will be the same, but the characteristics will be different, the logarithm of 2 being 0.3010300; that of 20. . . 1.3010300; of 200. . . 2.3010300; that of 2000. . . 3.3010300; in like manner the decimal part of the logarithm of 4, of 40, of 400, &c. will be alike, but the characteristics will be different; and so of other numbers. The following example will serve to illustrate and confirm this doctrine.

The log. of 86750, *is* 4.947519.
The log. of 8675.0, *is* 3.947519.
The log. of 867.50, *is* 2.947519.
The log. of 86.750, *is* 1.947519.
The log. of 8.6750, *is* 0.947519.
The log. of .86750, *is* -1.947519.
The log. of .086750, *is* -2.947519.
&c. &c.

17. From what has been said on the subject, we derive a more accurate definition of logarithms than that given above, namely, *logarithms are the indices of the ratios of numbers to unity*; that is, if 1 be considered as the common consequent, the ratio of any number to it is expressed by the logarithm of that number.

18. Further, to explain the properties and uses of logarithms, it may be observed, that the logarithms being the indices of a series of numbers in Geometrical Progression, it evidently follows from the nature of both, that the multiplication of numbers is performed by adding together their logarithms; division of numbers, by subtracting the logarithm of the divisor from that of the dividend; involution of numbers, by multiplying the logarithm of the root into the logarithm or index of the power; and evolution of numbers, by dividing the logarithm of the given number by the logarithm or index of the root. We will resume the series of numbers and their logarithms, in order to illustrate each case by a suitable example.

0.	1.	2.	3.	4.	5,	&c. *logarithms.*
1.	10.	100.	1000.	10000.	100000,	&c. *numbers.*

First. Let it be required to multiply any two terms in the above series of numbers together, suppose 10 and 1000. Add their logarithms (viz. 1 and 3) together, and the sum will be 4, the logarithm of 10000, (or of 10×1000,) the product of the two proposed numbers.

Secondly. Let it be required to divide 1000 by 10.

From 3 the log. of 1000, take 1 the log of 10, and the remainder 2 is the log. of 100, (or of $1000 \div 10$,) the quotient of the proposed numbers.

Thirdly. Let 100 be involved to the second power.

Multiply 2 (the log. of 100) into 2, which is the index or logarithm of the second power, and the product 4 is the log. of 10000, or of 100^2,) the power required.

Lastly. To extract the cube root of 1000.

Divide the log. of 1000, viz. 3, by the index 3 of the cube, and the quotient 1 is the log. of 10, (or of $\sqrt[3]{1000}$,) the root required.

19. To sum up the whole, by way of brief recapitulation. It has been shewn, 1. That logarithms are the indices or exponents of a series of terms in geometrical progression. 2. That every logarithm consists of two parts, a characteristic, and a decimal. 3. That the characteristic marks the step, that is, *what power of* 10 the number belonging to the logarithm is in; and the decimal shews the exact point which it occupies in that step, so that both together indicate *precisely* the corresponding number. 4. That the characteristic of a whole number is affirmative, or +, and that of a fraction negative, or −. 5. That addition and subtraction of logarithms respectively perform multiplication and division of their correspondent numbers; and that multiplication and division of the logarithms perform respectively involution and evolution of their numbers; and that herein consists their great value and usefulness, viz. by diminishing the time and labour otherwise necessary to the performance of tedious calculations.

20. There still remains one grand difficulty. It is sufficiently clear that 0, 1, 2, &c. are the logarithms of 1, 10, 100, &c. and that the doctrine above stated applies well enough to this series of geometricals; but how can logarithms, according to this system, be adapted to all the intermediate numbers, which are certainly not in geometrical but in arithmetical progression; for in order to be generally useful in calculations, every number ought to have its own particular logarithm? This we have promised to explain; and to shew, that logarithms may be found for all the numbers, agreeably to

the above-mentioned system, if not to absolute exactness, sufficiently near it for every practical purpose: in order to which we will once more resume the two series, viz.

0. 1. 2. 3. 4, &c. *logarithms.*
1. 10. 100. 1000. 10000, &c. *numbers.*

Now in order to find the logarithm of any intermediate number, situated between any two terms of the geometrical series, 1, 10, 100, &c.

1. Call the two terms between which the intermediate number lies, *the extremes.*

2. Between the two extremes, find a geometrical mean proportional, by multiplying them together, and extracting the square root of the product.

3. Take the logarithms of the two extremes, add them together, and divide the sum by 2; the quotient will be the arithmetical mean between the logarithms of the two extremes, and consequently the logarithm of the geometrical mean found above.

4. Take this geometrical mean, and the extreme between which the given number is, and find a geometrical mean between them as before.

5. Find an arithmetical mean between the logarithms of the said geometrical mean and extreme, and it will be the logarithm of the last found geometrical mean.

6. Proceed in this manner, by continually finding geometrical means between the two numbers adjacent to that whose logarithm is sought, until a mean proportional is obtained, indefinitely near the given number.

7. Find as many arithmetical means in the same order; the last will be the logarithm of the last geometrical mean, or of the given number indefinitely near. The following example, taken from Mr. Bonnycastle's Algebra, will fully illustrate this rule.

Let it be required to find the logarithm of 9.

First. *Because* 9 *lies between* 1 *and* 10, *therefore by the rule,* $\sqrt{1 \times 10} = \sqrt{10} = 3.1622777 =$ *the geometrical mean between* 1 *and* 10.

Now the log. of 1 *is* 0, *and the log. of* 10 *is* 1; *therefore* $\frac{0+1}{2} = \frac{1}{2} = .5 =$ *the arithmetical mean between* 0 *and* 1. *Whence the logarithm of* 3.1622777 *is* .5.

Secondly. The given number 9 *lies between* 3.1622777 *and* 10; *therefore* $\sqrt{3.1622777 \times 10} = \sqrt{31.622777} = 5.6234132 =$ *the geometrical mean between* 3.1622777 *and* 10.

Now the log. of 3.1622777 *is* .5, *and the log. of* 10 *is* 1; *therefore* $\frac{.5+1}{2} = \frac{1.5}{2} = .75 =$ *the arithmetical mean between* .5 *and* 1.

Whence the logarithm of 5.6234132 *is* .75.

Thirdly. The given number 9 *lies between* 5.6234132 *and* 10; *therefore* $\sqrt{5.6234132 \times 10} = 7.4989421 =$ *the geometrical mean.*

Now the log. of 5.6234132 *is* .75, *amd the log. of* 10 *is* 1; *therefore* $\frac{.75+1}{2} = .875 =$ *the arithmetical mean. Whence the logarithm of* 7.4989421 *is* .875.

Fourthly. The given number 9 *lies between* 7.4989421 *and* 10; *therefore* $\sqrt{7.4989421 \times 10} = 8.6596431 =$ *the geometrical mean.*

Now the log. of 7.4989421 *is* .875, *and the log. of* 10 *is* 1; *therefore* $\frac{.875+1}{2} = .9375 =$ *the arithmetical mean. Whence the logarithm of* 8.6596431 *is* .9375.

Fifthly. The given number 9 *lies between* 8.6596431 *and* 10; *therefore* $\sqrt{8.6596431 \times 10} = 9.3057204 =$ *the geometrical mean.*

Now the log. of 8.6596431 *is* .9375, *and the log. of* 10

is 1; *therefore* $\frac{.9375 + 1}{2} = .96875 =$ *the arithmetical mean. Whence the logarithm of* 9.3057204 *is* .96875.

Sixthly. The given number 9 *lies between* 8.6596431 *and* 9.3057204; *therefore* $\sqrt{8.6596431 \times 9.3057204} = 8.9768713$ = *the geometrical mean.*

Now the log. of 8.6596431 *is* .9375, *and the log. of* 9.3057204 *is* .96875; *therefore* $\frac{.9375 + .96875}{2} = .953125$, *which is the logarithm of* 8.9768713. *And proceeding in this manner, after* 25 *operations, the logarithm of* 8.9999998 *is found to be* .9542425, *which may be taken for the logarithm of* 9, *since* 8.9999998 *differs from* 9 *by only one five millionth part of an unit, an error too small to be of any consequence in practice.*

21. Exactly in the same manner the logarithms of any other numbers may be computed: but although this method is exceedingly obvious, and well adapted to explain the theory, the labour of making logarithms by it is excessive, as appears from the above example; and therefore other methods of computing them, derived from the nature of curves, infinite series, fluxions, &c. have in general been preferred, whereby much time and labour are saved.

22. Having computed the logarithms of a few of the *prime* numbers, those of their *composites, powers, roots,* &c. may be readily obtained; the former by addition only, and the two latter by an easy process in multiplication or division; and thus innumerable other logarithms may be obtained. It will not be amiss to give an example or two of this.

Let us suppose the logarithm of 2, and that of 3, to be found by the above, or any other method; the log. of 2 being 0.3010300, and that of 3 being 0.4771213.

Now suppose it be required to find the log. of 6, which is the product of 2 and 3.

Since addition of the logarithms produces multiplication of their correspondent numbers, we have only to add the logarithms of 2 and 3 together.

Thus, the log. of 2 = 0.3010300
the log. of 3 = 0.4771213
their sum = 0.7781513 = *the log. of* 6.

Let it be required to find the logarithm of 18.

Thus, add the log. of 6 = 0.7781513
to the log. of 3 = 0.4771213
their sum = 1.2552726 = *the log. of* 18.

To find the logarithm of 128, which is the seventh power of 2.

Multiply the log. of 2 = 0.3010300
By the index of the 7th power = 7
the log. of 128 = 2.1072100

To find the logarithm of the square of 3, viz. of 9.

Multiply the log. of 3 = 0.4771213
By the index of the square = 2
the log. of $\overline{3}|^2$ *or* 9 = 0.9542426

To find the log. of the 100th root of 2. Divide the log. of 2 by 100. *thus* 100)0.3010300(
the log. required = 0.0030103

In like manner the logarithms of all the powers and roots of 2 and 3 may be found; and likewise the products, quotients, powers, roots, &c. of the former, and so on, from the logarithms of 2 and 3 being given: hence the logarithms of the prime numbers being known, those of the other numbers may be derived by methods similar to those above.

23. It seemed necessary to anticipate thus much of the following part of the subject, to shew that loga-

rithms for all numbers are derived solely from the principle of an arithmetical progression, applied to a geometrical one, which has been stated and explained above. It is likewise proper, that the learner should know how to examine the accuracy of any logarithm; but it would not be worth his while to undertake the laborious task of computing a system of those numbers, since the tables of logarithms already in print are sufficiently accurate and extensive for every practical purpose.

24. The logarithms are placed in tables opposite their correspondent natural numbers, for the convenience of practice; so that a number being given, its logarithm may be readily found, and in like manner the number may be found from its logarithm being given. The following are the most usual plan and arrangement of the tables.—A page is divided from top to bottom into eleven columns, marked at top and bottom, N, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The left hand column, marked N, contains the four left hand figures of the natural number, to which for the right hand figure the proper one from the top or bottom must be taken; the remaining ten columns contain logarithms.

25. *To find the logarithm of a natural number consisting of four figures.* Look for the proposed number in the column marked N, and in a line with it in the next column stands its logarithm.

26. *To find the logarithm of a number consisting of five figures.* Look for the first four figures in the column marked N, and for the fifth figure at top or bottom; then in a line with the former, and in the same column with the latter, stands the proper logarithm. Thus, to find the logarithm of 2345; find this number in the column marked N, and opposite thereto, in the next column, stands the logarithm .3701428, which is the decimal part

only; the characteristic in this and every other instance being left for the operator to supply. Thus in the present example, if the given number 2345 be a whole number, 3 must be prefixed as a characteristic to the logarithm; if the last figure 5 be a decimal, 2 must be prefixed; if there are two decimals, 1 must be prefixed; if three decimals, 0; if the whole be a decimal, −1, &c.

To find the logarithm of 66534. Find 6653 in the column marked N, and 4 at the top; then in the column under 4, and level with 6653, stands .8230436; to which prefixing 4 for a characteristic, the required logarithm is 4.8230436.

In like manner, *the logarithm of*	4056 *is*	3.6080979.
That of	391 *is*	2.5921768.
That of	3.366 *is*	0.5271141.
That of	63.519 *is*	1.8029037.
That of	.88526 *is*	−1.9470708.

The characteristic in each of these, and in every other instance, being *one less than the number of integral places in the given natural number.*

26. *To find the natural number belonging to a logarithm.* Look for the logarithm in the columns marked 0, 1, 2, &c. and having found it, the number standing opposite, with the figure at the top of the column subjoined, will be the number; then mark off from this number as many places of whole numbers, as are equal to *one more than the index of the given logarithm,* and it will be the number required.

Thus, to find the natural number belonging to the logarithm 2.8230436, look in the columns marked 0, 1, 2, &c. for the decimal part only of this logarithm, (rejecting the index,) and having found it, the number opposite, in the column marked N, is 6653, and the figure at the head

of the column containing the given logarithm, is 4, which must be subjoined to the above four figures: and since 2 is the index of the given logarithm, three figures of this number must be pointed off for whole numbers; whence the natural number agreeing with the above logarithm is 665.34, as was required.

In like manner, *the natural number belonging to the logarithm* 1.7889104 *is* 61.505.

That belonging to the log. 3.9181562 *is* 8282.4.

That belonging to the log. 0.5410798 *is* 3.476.

That belonging to the log. 2.1682617 *is* 147.32.

That belonging to the log. -2.9187483 *is* .082937.

27. In the four former examples, the places of whole numbers pointed off are *one more* than the index of the respective logarithm; in the latter example, the index -2, shews that the left hand figure (viz. 8.) of its number must stand in the second place *below* units; a cipher must therefore be placed before it, and the whole will be a decimal.

There are frequently two additional columns in the tables, one for the differences of every two adjacent logarithms, and the other for the proportional parts of those differences; each difference being divided into nine parts in the ratio of the numbers 1, 2, 3, &c. to 9, for the purpose of finding the logarithm of any number, containing one or two places more than the numbers in the tables consist of, and likewise the number corresponding to any logarithm between two adjacent ones in the tables. Ample directions for these purposes are given with every collection of tables *.

* Thus, *to find the logarithm of a number, consisting of six figures.* Find the decimal part of the logarithm for the first five figures, and take the difference between that, and the next greater logarithm. Find the difference in the column marked D, then under that difference in the column marked *pts*,

and against the figure occupying the sixth place, stands the part which must be added to the logarithm found.

To find the logarithm for seven figures. Find the logarithm for the first six, as before; then divide the number corresponding to the seventh figure (in the column of *pts* marked D,) by 10; add the quotient to the decimal part of the logarithm for six figures, observing to place the first figure on the right, in the eighth place of the logarithm.

To find a number to six, seven, or more figures, answering to any given logarithm. From the given logarithm subtract the next less; add as many ciphers to the right of the difference, as there are additional figures required; divide this quantity by the difference between the next greater and next less than the given logarithm; and the quotient will be the figures required.

And by a converse process, numbers consisting of six, seven, or eight places, answering to any intermediate logarithm, may be readily found. See *Vince's Trigonometry, Hutton's Mathematical Tables*, pp. 131, 132, 133, and 134. *second edit.*

LOGARITHMICAL ARITHMETIC.

28. Logarithmical Arithmetic teaches to perform arithmetical operations, by means of logarithms previously computed and arranged in tables for use.

MULTIPLICATION BY LOGARITHMS.

29. *When the indices of the logarithms are affirmative, or +.*

RULE I. Seek in the table the logarithms of the factors, place them one under another, and add them together; their sum will be the logarithm of the product.

II. Seek this logarithm in the table, and the natural number answering to it will be the product required [p].

EXAMPLES.

1. Multiply 200 and 12 together.

OPERATION.

The log. of 200 = 2.3010300
The log. of 12 = 1.0791812
The product 2400 . . . 3.3802112

Explanation.

I first find the logarithms of 200 and 12, prefixing to that of the former 2 for a characteristic, and to that of the latter 1; I place the logarithms one under the other, add them together, and then look for the decimal part of their sum (viz. .3802112) in the table among the logarithms, opposite which, in the column marked N, I find 2400, which is the product, and 3 being the characteristic, I mark off 4 places for whole numbers.

2. Multiply 15.27. by 3.172.

OPERATION.

The log. of 15.27 = 1.1838390
The log. of 3.172 = 0.5013332
The product 48.437 = 1.6851722

Explanation.

Having looked out the logarithms, placed them under each other with their proper indices, and added them together, I find the number 48437 is the nearest in the table, which answers to their sum; from this I mark off 2 places of whole numbers, because the index of the sum is 1.

3. Multiply 1.2345 . . . 20.517, and 5.4321 together.

The log. of 1.2345 = 0.0914911
The log. of 20.517 = 1.3111139
The log. of 5.4321 = 0.7349678
The product = 137.27 = 2.1375728

---

[p] The truth of this rule is plain from the nature of logarithms, which has been fully explained in the Introduction.

4. Multiply 9.27 … 1.053 … 13.954, and 2.3456 together.

The log. of 9.27	=	0.9670797
The log. of 1.053	=	0.0224284
The log. of 13.954	=	1.1446987
The log. of 2.3456	=	0.3702540
The product = 319.49	=	2.5044608

30. *When any of the indices are negative, or* —.

RULE I. Find the sum of the decimals as before, then add what is carried, and the affirmative indices into one sum, and the negative indices into another.

II. Subtract the less of these sums from the greater, and to the remainder prefix the sign of the greater, and it will be the index to be prefixed to the decimal part of the sum [q].

5. Multiply 18.32 … 2.405, and .61245 together.

OPERATION.

The log. of 18.32	=	1.2629255
The log. of 2.405	=	0.3811151
The log. of .61245	=	—1.7870706
The product = 26.984	=	1.4311112

Explanation.

The 1 carried from the decimal part, is added to the index 1, making 2: then the —1 is taken from 2, and the remainder 1 is the proper index.

6. Multiply 1584.3 and .05637 together.

OPERATION.

Log. of 1584.3	=	3.1998374
Log. of .05637	=	—2.7510480
Product 89.307	=	1.9508854

Explanation.

There being nothing to carry from the decimal, I have only to subtract 2 from 3, and place the remainder 1 for an index.

7. Multiply .703 … .0918, and 47.345 together.

OPERATION.

Log. of .703	=	—1.8469553
Log. of .0918	=	—2.9628427
Log. of 47.345	=	1.6752741
Product 3.0554	=	0.4850721

Explanation.

Here 2 being carried from the decimals, the sum of the affirmative indices is equal to that of the negative ones, each being 3, whence 0 is the index to be supplied.

---

[q] We have before shewn, that the decimal part of a logarithm is *always* affirmative, and therefore the number carried from that decimal will evidently be affirmative. The reason why the sum of two numbers, having different signs, is found by subtraction, is explained in the notes on Addition of Algebra.

8. **Required the product of .3817 . . . .025913 and .998?**

OPERATION.

Log. of .3817	=	−1.5817222
Log. of .025913	=	−2.4135177
Log. of .998	=	−1.9991305
Product .0098712	=	−3.9943704

Explanation.

The sum of the indices is 4, and all being negative, I subtract the 1 carried, and —3 remains to prefix.

9. **Multiply 23.45 by 5.432.** *Prod.* 127.38.

10. **Multiply 4.9053 by 10.56.** *Prod.* 51.8.

11. **Multiply 1.7254 by .17254.** *Prod.* .2977.

12. **Multiply .32 . . . 4.08, and .12 together.** *Prod.* .15667.

13. **Multiply 1237 . . . 12.37, and .1237 together.** *Prod.* 1892.8.

14. **Multiply .03 . . . .004 . . . 157.8, and .0006 together.** *Prod.* 000011362.

31. DIVISION BY LOGARITHMS.

RULE I. From the decimal part of the logarithm of the dividend, subtract the decimal part of the logarithm of the divisor.

II. Change the index of the divisor, if it be affirmative, to negative, and if it be negative, to affirmative.

III. If after this change the indices have like signs, add them together, and prefix their sum, with its proper sign, to the decimal.

IV. If the indices after this change have unlike signs, take their difference, and prefix the remainder, with the sign of the greater, to the decimal.

V. Observe, that when 1 is carried from the decimal, it must be added to the index of the divisor, if affirmative, but subtracted, if negative; and this must be done in both cases before the index of the divisor is changed.

EXAMPLES.

1. Divide 8351.6 by 19.23.

OPERATION.

Log. of 8351.6	=	3.9217697
Log. of 19.23	=	1.2839793
The quotient = 434.3	=	2.6377904

Explanation.

Having placed the log. of the divisor under that of the dividend, I subtract the former from the latter; the remainder being the log. of the quotient, I look it out in the table, and find its natural number to be 434.3, which is the quotient required.

2. What is the quotient of 28.5 divided by 301.23?

OPERATION.

Log. of 28.5	=	1.4548449
Log. of 301.23	=	2.4788982
Quotient = .094612 =		−2.9759467

Explanation.

Having subtracted the decimals, there is 1 to carry to the 2, which makes 3; changing the sign, it becomes −3; I then take the difference of 3 and 1, which is 2, and prefix to it the sign, — (because 3 is negative,) —2 therefore is the index to be prefixed to the decimal remainder.

3. Divide .05432 by .2345.

OPERATION.

Log. of .05432	=	−2.7349598
Log. of .2345	=	−1.3701428
Quotient .23164	=	−1.3648170

Explanation.

There being nothing to carry from the decimal, I have only to change the sign of −1 to +1, to take the difference of 2 and 1, to prefix the sign − to the 1 remainder, and make it the index.

4. Divide 1.908 by .00095.

OPERATION.

Log. of 1.908	=	0.2805784
Log. of .00095	=	−4.9777236
Quotient = 2008.4 =		3.3028548

Explanation.

The 1 carried from the decimal, taken from 4, leaves 3; the sign of which is changed from − to +; then + 3 added to 0, gives 3 for the index.

5. Divide 108 by 36. *Quotient* 3.
6. Divide 92.4 by 12. *Quotient* 7.7.
7. Divide 5.123 by 3.47. *Quotient* 1.4764.
8. Divide .67894 by 234.71. *Quotient* .0028926.
9. Divide 375.27 by 381.27. *Quotient* .98426.
10. Divide 73.106 by .8714. *Quotient* 83.895.

32. Division may be performed by using instead of the logarithm of the divisor, its *arithmetical complement* [1], whereby subtraction is changed to addition.

The arithmetical complement of a logarithm is what that logarithm wants of 10.

33. *To find the arithmetical complement of any logarithm.*

RULE. If the index of the given logarithm be affirmative, subtract it from 9; but if negative, add it to 9: then proceeding

[1] The use of the arithmetical complement was first introduced by Mr. Edmund Gunter, Professor of Astronomy at Gresham College, probably about the year 1620. See *Briggs's Arithmetica Logarithmica*, &c. *cap.* 15.

from left to right, subtract each of the decimal figures from 9, except the last or right hand figure, which must be subtracted from 10; the result is the arithmetical complement required*.

To find the arithmetical complement of the logarithm 2.7817544.

Here, beginning at the index 2, I subtract that and each of the figures (proceeding from left to right) from 9, except the right hand figure 4, which I subtract from 10, and the result is 7.2182456, the arithmetical complement required.

To find the arithmetical complement of the logarithm —1.8893017.

Beginning at the 1, *I* add *it* to 9, *(because it is negative,) and subtract the other figures from 9, proceeding from left to right as before, except the last figure 7, which I subtract from* 10. *The arithmetical complement therefore will be* 10.1106983.

In like manner, the arithmetical complement of the logarithm 1.5250448, will be 8.4749552.

The arithmetical complement of 0.8430458, will be 9.1569542.

That of the logarithm —1.9854714, will be 10.0145286.

That of —3.8653409, will be 12.1346591, &c. &c.

34. *To perform division by addition.*

RULE I. Under the logarithm of the dividend, write the arithmetical complement of the logarithm of the divisor, and add both together.

II. If the index of the sum be 10, or greater than 10, subtract 10 from it, and prefix the remainder (which will be affirmative) as an index to the decimal part of the sum.

III. If the index of the sum be less than 10, subtract it from 10, and to the remainder (which is negative) prefix the negative sign —, and place it as an index to the decimal part of the sum, as before.

* To divide by any number, or to multiply by its reciprocal, produces the same result; and therefore, to add any logarithm, or subtract its reciprocal, must evidently do the same: now the arithmetical complement of a logarithm is the reciprocal of that logarithm increased or diminished by 10, according as the index is negative or affirmative; which accounts for the subtraction or addition of 10, as in Art. 34.

11. **Divide 435 by 29.8.**

OPERATION.

Log. of the dividend 435 . . . . = 2.6384893
Arith. comp. log. of the divisor 29.8 = 8.5257837
The quotient = 14.597 . . . . = 1.1642730

Explanation.

The logarithm of the divisor 29.8 is 1.4742163; I find the arithmetical complement of this, and having placed it under the logarithm of the dividend 435, I add both together; and the index of the sum being 11, I subtract 10 from it, and place the 1 remainder for an index.

12. Divide .123 by 124.

OPERATION.

Log. of .123 . . . . = −1.0899051
Arith. co. log. 124 = 7.9065783
Quotient .00099193 = −4.9964834

Explanation.

The sum of the indices 7 − 1 is 6; this I subtract from 10, and prefixing the negative sign to the remainder 4, I place this for an index to the sum.

13. Divide 308.25 by 31.024.

Log. of 308.25 . . . . . . = 2.4889031
Arith. comp. log. 31.024 = 8.5083022
The quotient = 9.9359 = 0.9972053

14. Divide 1.4759 by 23.917.

Log. of 1.4759 . . . = . . . 0.1690569
Arith. co. log. 23.917 = . . . 8.6212933
Quotient .061709 . . . = −2.7903502

15. Divide .09876 by 98.76.

Log. of .09876 . . . = . . . −2.9945811
Arith. co. log. 98.76 = ... 8.0054189
Quotient = .001 . . . = . . . −3.0000000

16. Divide 1716 by 12. *Quotient* 143.
17. Divide 246.2 by 207.5. *Quotient* 1.1865.
18. Divide 8.3017 by .9012. *Quotient* 9.2119.
19. Divide 4.3213 by 5.1238. *Quotient* .84338.
20. Divide .09512 by 203.76. *Quotient* .00046672.

35. *The logarithm of a vulgar fraction is found by this rule.* From the logarithm of the numerator, subtract the logarithm of the denominator, (or add its arithmetical complement,) and the result will be the logarithm required.

If a mixed number be given, reduce it to an improper fraction, and then find its logarithm as above.

21. Required the logarithm of $\frac{3}{4}$?

From the log. of 3 = 0.4771213
Subtract the log. of 4 = 0.6020600

The log. of $\frac{3}{4}$ = -1.8750613

22. Required the logarithm of $3\frac{4}{5}$?

Thus $3\frac{4}{5} = \frac{19}{5}$

Then from the log. of 19 = 1.2787536
Subtract the log. of 5 = 0.6989700

The log. of $3\frac{4}{5}$ = 0.5797836

23. What is the logarithm of $\frac{8}{9}$? *Ans.* -1.9488475.

24. What is the logarithm of $12\frac{1}{2}$? *Ans.* 1.0969100.

36. PROPORTION BY LOGARITHMS.

RULE I. Prepare the terms as in the Rule of Three of Decimals, if they require it.

II. Place the terms (so reduced) in order, one under another, with the logarithm of each opposite to its respective term.

III. Add the logarithms of the two *multiplying* terms together, and from the sum subtract the logarithm of the *dividing* term; the remainder will be the logarithm of the answer.

IV. Or find the arithmetical complement of the *dividing* term, and add it and the logarithms of the two remaining terms together, observing to take the difference of 10 and the index, as in division; the result will be the logarithm of the answer as before.

EXAMPLES.

1. If the week's allowance for 5 seamen be 38*lb.* of biscuit, how many pounds will a ship's company of 224 men consume in the same time?

Operation.

First method.		*Second method.*	
As 5 *men*. . . . . . . *its log.* =	0.6989700	*Arith. co.* =	9.3010300
To 38 *lb.* . . . . . . *its log.* =	1.5797836	*Logar.* =	1.5797836
So are 224 *men.* . *its log.* =	2.3502480	*Logar.* =	2.3502480
From	3.9300316		
Subtract	0.6989700		
To 1702.4 *pounds* = ...	3.2310616	*Sum* =	3.2310616

Explanation.

Having stated the question according to the rule, I find that it belongs to the direct rule, and therefore the *first* is the dividing term; whence, under the first method, I add the logarithms of the first and second terms together, and subtract that of the first from the sum. By the second method, I find the arithmetical complement of the first term, and the logarithms of the two other terms, and add all the three together, subtracting 10 from the index of the sum; the result by both methods is the same, namely, the logarithm of the answer, which is of the same name with the second term, viz. pounds.

2. If 7 men can perform a piece of work in 54 days, in how many days would 23 men accomplish the same?

Operation.

First method.		*Second method.*	
As 7 *men*. . . . . *log.* =	0.8450980	*Log.* =	0.8450980
To 54 *days*. . . . . . . *log.* =	1.7323938	*Log.* =	1.7323938
So are 23 *men*. . . . . *log.* =	1.3617278	*Arith. co.* =	8.6382722
From sum of 1*st and* 2*nd* =	2.5774918		
Take the third . . . . =	1.3617278		
To 16.435 *days* . . . . . . =	1.2157640	*Sum* =	1.2157640

Explanation.

This example evidently belongs to the inverse rule, I therefore add the logs. of the first and second terms together; and from the sum subtract the log. of the third, according to the 1st method, or add its arithmetical complement, according to the 2nd.

3. If 3½*lb.* of tea cost 1*l.* 7*s.* 6*d.* what is the value of 2*cwt.* 3*qr.* 4*lb.*?

Operation.

First method.

As 3½*lb.* = .03125*cwt.* . . . . *log.* =	−2.4948500
To 1*l.* 7*s.* 6*d.* = 1.375*l.* . . . . *log.* =	0.1383027
So are 2*cwt.* 3*qr.* 4*lb.* = 2.7857*cwt.* . *log.* =	0.4449343
From	0.5832370
Subtract	−2.4948500
To 122.57*l.* = 122*l.* 11*s.* 4*d.*¾ = ...	2.0883870

Second method.

Ar. co.	=	11.5051500
Log.	=	0.1383027
Log.	=	0.4449343
Sum	=	2.0883870 *as before.*

Explanation.

The first term, and the qrs. and lbs. in the third, are reduced to decimals of a cwt.; and the shillings and pence in the second, to the decimal of a pound; after which the operation proceeds exactly as before.

4. If $3\frac{7}{8}$ yards of lace sell for 19*s.* 6*d.*$\frac{1}{2}$, what is the value of 17 pieces, each 16*yds.* 3*qr.* 2*na.*?

OPERATION.

First method.

As $3\frac{7}{8}$ *yd.* = 3.875*yd.*	*log.* =	0.5882717
To 19*s.* 6*d.*$\frac{1}{2}$ = .97708*l.*	*log.* =	-1.9899301
So are { 17 *pieces.*	*log.* =	1.2304489
{ *each* 16*y.* 3*q.* 2*n.* = 16.875	*log.* =	1.2272438
	From	2.4476228
	Subtract	0.5882717
To 72.335*l.* = 72*l.* 6*s.* 8*d.*$\frac{1}{2}$. =		1.8593511

Second method.

Ar. co.	=	9.4117283
Log.	=	-1.9899301
{ *Log.*	=	1.2304489
{ *Log.*		1.2272438
Sum	=	1.8593511 *as before.*

Explanation.

This operation at first sight appears to have four terms, but the two latter evidently constitute but one term, namely, the third; the pieces and yards are multiplied together by adding their logarithms.

5. Find a fourth proportional to 123.4 . . . 43.21, and 1.

As 123.4 . . . . *log.* =	2.0913152	*Arith. co.* =	7.9086848
To 43.21 . . . . *log.* =	1.6355843	*Log.* . . . =	1.6355843
So is 1 . . . . . *log.* =	0.0000000	*Log.* . . . =	0.0000000
To .35016 = ...	-1.5442691	*Sum* = .	-1.5442691

6. If a pipe of Madeira cost 109*l.* 10*s.* what must be given for 1*hhd.* 25 *gallons* of the same?

As 1 *pipe* . . . . *log.*	=	0.0000000
To 109*l.* 10*s.* = 109.5*l.* . . *log.*	=	2.0394141
So are 1*hhd.* 25 *gal.* = .69841 *pipe*	=	−1.8441104
To 76.476*l.* = 76*l.* 9*s.* 6*d.*	=	1.8835245

7. If 20*lb.* of sugar cost 1*l.* 2*s.* 6*d.* what will 56*lb.* cost? *Answer* 3*l.* 3*s.*

8. Required a fourth proportional to the three given numbers, 11, 12, and 20? *Ans.* 21.818, &c.

9. If $\frac{3}{4}$ of a gallon of wine cost $\frac{5}{8}l.$ what will $1\frac{4}{5}$ gallon cost? *Ans.* 1*l.* 10*s.*

10. A person receives 82*l.* 7*s.* 6*d.* per year, how much is that per day? *Ans.* 4*s.* 6*d.*

11. Required a third proportional to 123 and 234? *Answer* 445.17, &c.

12. If A lends B 332*l.* for 7 months, what sum ought B to lend A for 20 months, to discharge the obligation? *Answer* 116*l.* 4*s.*

37. INVOLUTION BY LOGARITHMS.

RULE I. Multiply the logarithm of the number to be involved, by the index of the proposed power, and the product will be the logarithm of the power.

II. If the index of the logarithm be negative, the product of the index will be negative; and since what is carried is affirmative, the difference of these two must be taken, and the sign of greater prefixed to it, for the index of the logarithm of the proposed power.

EXAMPLES.

1. Involve 12.916 to the second power.

OPERATION.

Logarithm of 12.916	=	1.1111280
Index of the 2*nd power*	=	2
Second power = 166.82	=	2.2222560

Explanation.

Having found the logarithm of the given number, I multiply it by 2, the index of the second power; the product is the logarithm of 166.82, the power required.

2. Involve .2037 to the third power.

OPERATION.

Log. of .2037	=	−1.3089910
Index of the 3*d power*	=	3
Third power = .0084523 =		−3.9269730

Explanation.

Here the index being negative, and nothing to carry from the decimal, the product of 3 into −1 will be wholly negative.

3. Involve .0421 to the thirtieth power.

OPERATION.

Log. of .0421	=	− 2.6242821
Index of the 30*th power*	=	30
Pow. .[41]53513	=	− 42.7284630

Explanation.

Here 18 carried from the decimal is to be subtracted from −60, the remainder −42 is the index; .[41] shews that 41 ciphers are to be prefixed to the number 53513.

4. Involve .1021 to the 365th power.

OPERATION.

Log. of .1021 =		−1.0090257
Index of 365*th power*	=	365
		451285
		541542
		270771
Product of the decimals =		3.2943805
Prod. of integers		−365
Power [361]19696 =		−362.2943805

Explanation.

In cases like the present, it is best to multiply the decimals and the integers separately, making two operations; we have here multiplied the decimals first, and afterwards the −1, then subtracted the index of the former product from the latter; the included number [361] shews that so many ciphers are to be prefixed to complete the decimal.

5. Involve 7.004 to the second power. *Power* 49.056.
6. Involve 23.123 to the third power. *Power* 12363.2.
7. Involve 1.012 to the twentieth power. *Power* 1.26943.
8. Involve .3128 to the eleventh power. *Power* .000002805.
9. Involve 7.1635 to the 1.2345 power. *Power* 11.3672.

EVOLUTION BY LOGARITHMS.

38. *When the index of the logarithm is affirmative.*

RULE. Divide the given logarithm by the number denoting the root, and the quotient will be the logarithm of the root required.

EXAMPLES.

1. Extract the square root of 26.725.

OPERATION.

$$\begin{array}{r} 2)\underline{1.4269177} \\ \underline{0.7134588} \end{array} \; Root = 5.1696.$$

Explanation.
I divide the log. of the given number by 2, the number denoting the square root; the quotient is the logarithm of 5.1696, the root required.

2. Extract the cube root of 5182.9.

Logarithm.

$$\begin{array}{r} 3)\underline{3.7145728} \\ \underline{1.2381909} \end{array} \; Root = 17.3058.$$

3. Extract the 12th root of 98765.

Logarithm.

$$\begin{array}{r} 12)\underline{4.9946031} \\ \underline{0.4162169} \end{array} \; Root = 2.60745.$$

39. *When the index of the logarithm is negative.*

RULE. If the index of the logarithm be divisible without remainder, divide as before, making the index of the quotient negative; but if not, borrow as many as will make it exactly divisible, carrying the number borrowed as so many tens to the left hand decimal place, and proceed as before; observing to make the index of the quotient negative.

4. Required the square root of .03974?

OPERATION.

Logarithm.

$$\begin{array}{r} 2)\underline{-2.5992279} \\ \underline{-1.2996139} \end{array} \; Root = .19935.$$

Explanation.
The index -2 being exactly divisible by 2, I divide as before, making the index 1 of the quotient negative.

5. What is the cube root of .482?

OPERATION.

Logarithm.

$$\begin{array}{r} 3)\underline{-1.6830470} \\ \underline{-1.8943490} \end{array} \; Root = .78406.$$

Explanation.
Since -1 is not divisible by 3, I borrow 2, and then divide, making the quotient 1 negative; I then carry the 2 borrowed as 20 to the 6, and proceed exactly as before.

6. Required the 12th root of .0012834?

OPERATION.

Logarithm.

12) -3.1083620

-1.7590301 *Root* = .574156.

Explanation.

Here I borrow 9 to the index, and after dividing it, I carry 90 to the 1, and proceed.

7. Extract the fifth root of .0000006789.

OPERATION.

Logarithm.

5) -7.8318058

-2.7663611 *Root* = .058393

Explanation.

I borrow 3 to make the index exactly divisible, put down the quotient -2, carry 30 to the decimal, and proceed as before.

8. What is the square root of 961? *Root* 31.
9. Required the cube root of 1.341? *Root* 1.1?
10. Extract the fourth root of 381.25. *Root* 4.41878.
11. Required the square root of .026131? *Root* .16165.
12. Required the cube root of .70017? *Root* .887977.
13. Extract the fourth root of .61042. *Root* .88391.
14. What is the tenth root of 1087.4? *Root* 2.01205.
15. Extract the eleventh root of 99999. *Root* 2.84803.
16. What is the 100th root of 200? *Root* 1.05441.
17. What is the cube root of .00000027? *Root* .0064633.

40. ***When the root is denoted by a fraction, the numerator of which is greater than unity.***

RULE. Multiply the logarithm of the given number by the numerator of the fraction, and divide the product by the denominator.

Or, Divide the logarithm by the denominator, and multiply the quotient by the numerator.

Or, Reduce the fraction to a decimal, and multiply the logarithm of the given number by it: the result in each case will be the logarithm of the root required.

18. Required the value of $\overline{7845}|^{\frac{3}{4}}$.

First method.

Log. of 7845 = 3.8945929

3

4)11.6837787

Root 835.575 . . . 2.9209446 . . . *log.* . . . 2.9209446

Second method.

4)3.8945929

0.9736482

3

2.9209446

Third method.

$$\frac{3}{4} = .75 \quad \textit{Then} \quad \begin{array}{r} 3.8945929 \\ .75 \\ \hline 194729645 \\ 272621503 \\ \hline \end{array}$$

$$\textit{Root} = 835.575 \ldots \ 2.9209446|75$$

19. Required the value of $\overline{.13579}|^{\frac{7}{8}}$?

First method.

$$\begin{array}{rr} \textit{Log. of } .13579 = & -1.1328678 \\ & 7 \\ \hline 8) & -7.9300746 \\ \hline \textit{Root} = .174284 \ldots & -1.2412593 \ldots \textit{log.} \\ \hline \end{array}$$

Second method.

$$\begin{array}{r} 8)-1.1328678 \\ \hline -1.8916084 \\ 7 \\ \hline \ldots -1.2412588 \\ \hline \end{array}$$

Third method.

$$\frac{7}{8} = .875 \quad \textit{Then} \quad \begin{array}{r} -1.1328678 \\ .875 \\ \hline .6643390 \\ 9300746 \\ 10629424 \\ \hline \end{array}$$

$$\begin{array}{rr} \textit{Prod. of the decimals} \times .875 = & .1162593250 \\ \textit{Prod. of} -1 \times .875 = & -875 \\ \hline \textit{Root} = .174284 \ldots & -1.2412593|250 \\ \hline \end{array}$$

Explanation.

The two right hand figures in the 1st and 2nd methods do not agree, owing to the remainder in the first line of the latter. In the 3d method I multiply, first, the decimals only, and afterwards the -1; then because the latter product is negative, I substract it from the former, putting down -1, for the 10 borrowed at the left hand decimal figure: this method is always followed in similar cases.

20. What is the value of $\overline{123}|^{\frac{2}{3}}$? *Ans.* 24.7326.

21. What is the value of $\overline{80}|^{\frac{3}{5}}$? *Ans.* 13.8629.

22. Required the value of $\overline{2.152}|^{\frac{7}{9}}$? *Ans.* 1.815.

23. Required the value of $\overline{.000435}|^{\frac{5}{6}}$? *Ans.* .0015803.

24. Find the root of $\overline{6.07}|^{2\frac{1}{2}}$, or its equal $\overline{6.07}|^{\frac{5}{2}}$. *Ans.* 90.7761.

41. PROMISCUOUS EXAMPLES FOR PRACTICE.

1. Find the value of $\dfrac{11.382 \times 2.345 \times 3.921}{12.845}$.

$$\textit{Thus, Add}\ldots\begin{cases} \textit{log. of } 11.382 = 1.0562186 \\ \textit{log. of } 2.345 = 0.3701428 \\ \textit{log. of } 3.921 = \underline{0.5933968} \end{cases}$$

From the sum = ... 2.0197582
Subtract log. of 12.845 ..... = 1.1087341
Remains log. of 8.1475 ..... = 0.9110241

2. Find $\overline{1.085}|^{2} \times \overline{.012}|^{3} \times \overline{194.57}|^{\frac{1}{2}} \div \overline{38.325}|^{\frac{1}{3}}$. *Answer* .0000084164.

3. Find $\overline{12.004 \times \overline{3.017}|^{2} \times \sqrt{4.21 \times \overline{.0334}|^{3}}}|^{4}$ *Ans.*

4. Find $70.92 \times 7.1234 \div 12.8 \times \overline{3.003}|^{3}$. *Ans.*

5. Find $\overline{9.0129}|^{5} \times \overline{9.817}|^{\frac{2}{3}} \div \overline{8.01}|^{\frac{4}{7}}$. *Ans.*

6. Required $\overline{37.123}|^{\frac{1}{3}} \div \overline{\overline{38.31}|^{\frac{5}{9}} \times \overline{101.23}|^{\frac{11}{12}}}$? *Ans.*

7. To find $3\frac{7}{8} \times \overline{2.34}|^{\frac{3}{5}} \times \overline{45.67}|^{\frac{2}{7}} \div \overline{12.123}|^{\frac{8}{9}}$. *Ans.*

8. What is $\overline{13.107}|^{2} + \overline{21.14}|^{\frac{1}{7}} - \overline{31.23}|^{\frac{2}{3}} - \overline{40.76}|^{\frac{3}{8}}$? *Ans.*

9. To find a mean proportional between the square of 10, and the cube root of 20.

10. If the side of a cube be 8, required the side of another cube exactly double the former?

11. To find a third proportional to 45 and double that number.

12. To find 5 mean proportionals between 24 and 25.

PART III.

ALGEBRA.

ALGEBRA is an universal method of reasoning on quantity by means of general characters. It is applied to the resolution of all kinds of problems wherein quantity is concerned; for which purpose it does not require that rules should be previously laid down, but teaches how to discover or invent them, and that by the force of reasoning, from a bare contemplation of the conditions and relations of the quantities, as expressed in the problem under consideration. Algebra likewise shews how to demonstrate or prove the rules, theorems, and conclusions, thus investigated.

It is impossible to do justice to this elegant and useful branch of learning by any description; all that is attempted in this place is to give the learner a general, although necessarily an inadequate, conception of the subject, which it is hoped will nevertheless be of service to him in his progress. The quantities which occur in Algebra are represented by the small alphabet, or other convenient symbols, which, standing for no particular value themselves, are made the arbitrary representatives of the quantities in question: this general mode of representation is attended with very great advantages; one of which is that the solution of a problem, by the general method of Algebra, furnishes an answer to every

particular problem of the kind that can possibly be proposed, by merely substituting the numbers concerned in the particular problem, in the place of their corresponding letters in the answer to the general one. Letters and symbols being made the representatives of quantities, are managed like numbers, and consequently like them are subject to the fundamental rules of common arithmetic; and their relations and operations are denoted by the same marks or signs.

When a problem is to be resolved, the first thing necessary to be done is to translate it out of common into algebraic language, by substituting letters for all the quantities concerned, both known and unknown; namely, by putting one of the initial letters for each known quantity, and final letters for the unknown ones, and expressing the conditions and relations of the quantities by their proper signs; this is *the composition:* when this is effected, we shall have an expression, wherein one or more quantities are declared equal to some other quantities; this expression is called an equation. Having made the composition, the next thing to be done is to find the value of the unknown quantities contained in the equations, which process is called *the resolution:* thus each unknown quantity must be disentangled from all known ones connected with it on the same side of the sign of equality, which is effected in each case by a process the contrary to that by which they are connected. A known quantity connected by *addition,* is taken away by *subtraction;* namely, by subtracting it from both sides of the equation; if it be connected by *subtraction,* it is taken away by *addition;* if by *multiplication,* it is taken away by *division,* &c. &c. always employing a contrary process: and thus the unknown quantity is at last found by itself on one side of the equation, and known ones

only on the other; thus the value of the unknown quantity is found, for (as the equation thus reduced implies) it is equal to the known ones, connected together according to the import of their signs.

Of the origin and early history of Algebra nothing is known. The Algebra of Diophantus, a Greek of Alexandria, who is supposed to have flourished about the second century after Christ, is the earliest work on the subject that has descended to us: in this work Diophantus has not given the principles and elements, but confined himself to questions depending on the most curious properties of numbers, which require considerable skill and address to resolve them; whence it is inferred, that Algebra must have been cultivated by the ancient Greeks, and had arrived at a considerable degree of perfection before the time of Diophantus. The destruction of the Alexandrian Library, A. D. 642, by which the sciences suffered an irrecoverable shock, most probably deprived us of many valuable writings, which would undoubtedly have thrown considerable light on the subject.

But although Diophantus was the earliest author that is known to have written on Algebra, whence it can hardly be supposed that the ancient Greeks were unacquainted with the science, it was not from them, but from the Arabians, that the knowledge of Algebra was acquired by the western nations. The Italians were the first Europeans who cultivated this branch of science. Leonard de Pisa, and several others of that country, are said to have possessed great knowledge in the various methods of resolving problems, as known among the Arabians. M. Bossut affirms, that Leonard de Pisa flourished as early as the beginning of the thirteenth century, and that from a manuscript of his, discovered and quoted by Cossali, it appears he understood

the method of solving not only cubic equations, but also those of higher powers, capable of being reduced to the second and third orders [a]. The Germans were acquainted with Algebra at an early period, as appears from a treatise on Plane and Spherical Trigonometry, by the famous Regiomontanus, written about the year 1464; wherein some of the problems concerning rectilineal triangles are accompanied with an algebraic solution [b].

The analytical works of Leonard de Pisa remaining in manuscript were scarcely known, even in Italy, so that Lucas De Burgo [c] is universally considered as the first who wrote professedly on Algebra in Europe, at least whose works were printed. His great work, entitled *Summa de Arithmetica et Geometria*, &c. was published in 1494 at Venice, and is considered as a very complete and masterly work on the sciences treated of, as they then stood. After naming several authors from whom he acquired the knowledge of the sciences, he proceeds to treat of Numbers figurate, odd, even, perfect, prime, composite, and many others; then of Numeration or Notation, Addition, Subtraction, Multiplication, Division, Progression, Evolution, &c. performing and proving his operations by various methods; by casting out the nines, sevens, &c. he extracts the square and cube roots, after the manner now in use, denoting a root by the initial R. He

[a] See, Origine, Transporto in Italia e primi Progressi in Essa del Algebra, &c. 1797. quoted in Bossut's General History of the Mathematics. London, 1803.

[b] See the introduction to Dr. Hutton's Mathematical Tables, p. 3.

[c] Lucas Paciolus, commonly called Lucas De Burgo, because he was born at Borgo San Sapocha in Tuscany, was a Cordelier, or Minorite Friar, and the first who occupied the mathematical professorship founded at Milan, by Lewis Sforsa, called the Black. He translated Euclid into Latin, or rather revised the translation of Campanus, which he enriched by many learned annotations. He wrote several treatises on Arithmetic, Algebra, Geometry, Perspective, Music, Architecture, &c. which were published between the years 1470 and 1500.

treats of Vulgar Fractions, the Rule of Three, Loss and Gain, with other rules used by merchants, in the same way we do: he then proceeds to Algebra, ascribing the invention to the Arabians; he shews the method then in use of denominating the powers and roots, and the necessary abbreviations required in practice. He treats of Proportions and Proportionalities, Arithmetical and Geometrical, accompanied with a copious collection of questions relating to numbers in continued proportion. Single and double Position are next unfolded by nearly the same method as at present; then follow the common operations of Algebra, proving that like signs give *plus*, and unlike *minus*, both in multiplication and division. He treats of the Extraction of Roots, Surds, simple and quadratic Equations, completing the Square and extracting the Root, all by the methods at present in use; he resolves Equations of the simple fourth power, and of the fourth combined with the second, treating the latter the same as quadratics. In the third case of quadratics, which has two positive roots, he uses both, but takes no notice of the negative roots which occur in both the other cases. He offers no solution of any other forms of affected Equations besides those above-mentioned, and says that no general rule for that purpose was then known. The remaining part of this work is on Geometry.

The state of Algebra, at the time of its reception into Europe, is supposed to be fully exhibited in this book: the solution of quadratics, restricted to the use of the affirmative roots, is the highest pitch to which the science is here carried; so that if the method of solving higher equations was known to the Arabs, as has been asserted, the Europeans did not learn it from them[d]. It

[d] The instance we have quoted from Bossut cannot be considered as an ex-

appears also, that the Algebraists of this period knew only how to solve numerical problems with one unknown quantity, and that they had no signs for either the quantities or operations, except abbreviations of the words themselves.

The publication of De Burgo's book seems to have been the means by which the knowledge of Algebra was first diffused through Italy, and other neighbouring countries, as we find many learned men, both there and in various parts of Germany, now began to study the science with so much success, that a very few years produced many learned Algebraists.

About the year 1505, Scipio Ferreus, Professor of Mathematics at Bononia, discovered the method of solving one case of Cubic Equations; but his method was most probably (agreeably to the fashion of those dark ages) retained as a secret. Thirty years after, Nicholas Tartalea[e], a Mathematician of Brescia, made the same discovery, with the addition of the rules for the two remaining cases; proposing, in like manner, to conceal the methods of investigation. The secret was nevertheless, after much difficulty, drawn from him by the address of Jerome Cardan[f], a celebrated Physician, Astrologer,

ception: Leonard De Pisa's tract on Cubics, &c. (if such a work really existed,) was probably a *single* manuscript, and totally unknown to the few Algebraists of that period. Every circumstance attending the discovery of the rules for cubics, which happened a few years after, adds weight to this conclusion.

[e] Tartalea was born at Brescia in Italy, towards the close of the 15th century: he was a very respectable teacher of the Mathematics, and published various works of merit on that subject, the chief of which was, *Trattato di Numeri et Misure*, fol. 1556. being a treatise on Arithmetic, Algebra, Geometry, &c. here are many of the curious particulars of the dispute between our author and Cardan. He published in 1543, at Venice, all the books of Euclid, with curious notes. He was the first author who treated of the flight and path of balls and shells, in a work entitled, *Nova Scientia Inventa*, 4to. published at Venice in 1537. an English translation of which, with notes and additions by Lucar, came out at London in 1588. Tartalea died about the year 1558.

[f] Hieronymus Cardanus was born at Pavia in Italy, in 1501. At twenty years

and Lecturer on the Mathematics at Milan; who after adding perjury to falsehood, dared to insert in a large work on the Mathematics, which he was then printing, those very rules which he had obtained from Tartalea under the most solemn promises, confirmed by an oath, of inviolable secresy.

Cardan, although a bad man, was indisputably the best algebraist of the age, and the improvements he introduced into the science were very considerable; especially in the discoveries he had drawn from Tartalea, deriving from them rules for the solution of all the forms of Cubics: he was well acquainted with all the real roots of Equations, both positive and negative; shewing that the even roots of positive quantities are either positive or negative, that the odd roots of negative quantities are real and negative, and that their even roots are impossible. He knew the number and nature of the roots of an Equation, as depending on the signs of the terms, and the magnitude and relation of the coefficients; that the number of positive roots is equal to the number of changes in the signs of the terms; that the coefficient of the second term is equal to the difference between the

of age he became a student at the University of Milan, and two years after explained Euclid. In 1524 he was admitted Master of Arts, and the year following Doctor of Physic; about 1533 he became Professor of Mathematics at Milan, where six years after he was received a member of the College of Physicians, and read public lectures on Medicine; he taught successively at Paria, Bologna, and Rome: at the latter he was admitted a member of the College of Physicians, and received a pension from the Pope, which he enjoyed till his death, which happened in 1575. Cardan was so great an adept in astrology, that the greatest personages in Europe had recourse to his skill; among these we find Edward VI. of England, whose nativity was calculated by our astrologer as he passed through London from Scotland, having been sent for there by the Archbishop of St. Andrews, to cure him of a dangerous disorder.

Cardan was the greatest, although the most eccentric and restless genius of his time; possessing splendid talents, accompanied with a wicked and depraved heart. The Lyons edition of his works printed in 1663, consists of no less than ten volumes folio.

positive and negative roots, and that consequently, when the second term is wanting, the sums of the negative and positive roots will be equal; that changing the signs of the even terms changes the signs of the roots; that the roots fail in pairs. He knew how to compose Equations with given roots, or to change them from one form to another, by taking away any intermediate term; he could extract the roots of such binomials as would admit of extraction: he knew all the difficulties attending the irreducible case of Cubics, and the attempts he made to solve it led him to the discovery of rules, whereby the roots may be approximated to, in all cases whatever: he frequently used the literal notation, expressing quantities by letters; treated fully on the transformation of Equations; and shewed how to apply Algebra to the solution of Geometrical Problems.

About the year 1540, Lewis Ferrari, the pupil of Cardan, discovered a rule for the solution of biquadratics, which the latter has demonstrated, explained, and exemplified, and given in his treatise on Algebra.

Tartalea's *Quesiti et Inventioni Diverse*, printed at Venice in 1546, and dedicated to King Henry VIII. of England, is chiefly remarkable for the account it gives of the invention of the above-mentioned rules for Cubic Equations, of the artful methods employed by Cardan to obtain them, and the quarrel which ensued. Tartalea was public Lecturer on the Mathematics at Venice. About this time Franciscus Maurolicus, Abbot[g] of Santa Maria del Porto, in Sicily, distinguished himself by his

[g] Franciscus Maurolicus was born at Messina in 1494: he was a great proficient in the Mathematics, which he taught with unbounded applause; to him we are indebted for the "*Tabula Benefica*," or Canon of Secants, and likewise an edition of the Spherics of Theodosius; Emendatio et Restitutio Conicorum Apollonii Pergæi; Archimedis Monumenta Omnia; Euclidis Phænomena, &c. he died in 1575.

great skill in the Mathematics; in particular, he cultivated a branch of analysis then but little known, namely, the summation of series; he gave theorems for summing the series of natural numbers, their squares, &c. for triangular and other figurate numbers, all remarkable for subtilty of invention, and simplicity of result.

Algebra seems to have been in a more advanced state about this time in Germany, than it was in Italy, and to have approached nearer to the modern method; although it does not appear that the Germans knew any thing of the rules for Cubics. The earliest writer of that country was Michael Stifelius [h], a Protestant minister, and an eminent mathematician; his chief work, entitled *Arithmetica Integra*, was published at Nuremberg in 1544: it is an excellent treatise on both Arithmetic and Algebra, and contains several ingenious inventions in both. In this work he introduces the characters +, −, and √, and the numeral exponents, both positive and negative, of powers, teaching the general use of exponents in the several operations on powers, as is practised at present. He understood the nature and use of Logarithms, although under another name; but it does not appear that he knew the use of fractional indices. He employed the capitals A, B, C, D, &c. to express unknown quantities, treated of quadratics in a more general manner than had been before done, and made various other improvements. John Scheubelius, Professor of Mathema-

[h] Stifelius was born at Eslingen in Germany, some time about the year 1490, and died at Jena in Thuringia in 1567; his improvements in Algebra are briefly mentioned above. Unfortunately he was not content with the credit of being a skilful mathematician, but wished to extend his fame by becoming a prophet; accordingly he predicted that the world would be at an end on a certain day in the year 1553: multitudes of his followers met him in the open air on the appointed day, but instead of being spectators of the awful event foretold, were witnesses only of the mortification and disappointment of the unfortunate prophet.

tics at Tubingen, in the Duchy of Wirtemberg, wrote several treatises on Arithmetic and Algebra, about the year 1550: he is the first algebraist who makes mention of Diophantus; most probably he knew nothing of the discoveries of Ferrari and Tartalea, as he takes no notice of Cubic Equations.

The first English writer whose works on Algebra were printed, was Dr. Robert Recorde[i], a learned physician and mathematician, who flourished under Edward VI. and Mary. He published a treatise on Arithmetic in 1552, entitled *The Ground of Arts*, a work much esteemed at that time, and which continued many years the standard in that branch of knowledge. In 1557 he sent abroad a second part, under the title of *Cos Ingenii, or the Whetstone of Witte;* this part treats of Algebra in the form of a dialogue: in his method he imitated the Germans Stifelius and Scheubelius, especially the latter, whom he sometimes quotes and copies. The first instance of the extraction of the roots of compound algebraic quantities occurs in this book, and here also are first introduced the terms *binomial* and *residual*, and the sign = of equality.

After Recorde's death, it appears that Algebra was not much cultivated in England for several years, inasmuch as John Dee[k], in his Preface to Billingsley's Euclid,

[i] Robert Recorde was born in Wales early in the 16th century; and about 1525 went to Oxford, where in 1531 he became fellow of All Souls College: making physic his profession, he repaired to Cambridge, where he was honoured with the degree of M. D. in 1545. He afterwards taught the Mathematics with great applause at Oxford, and probably next at London; he was physician to both the monarchs mentioned above, and the author of several mathematical treatises. He was confined for debt in the King's Bench Prison, where he died in the year 1558.

[k] John Dee was born in London in 1527; at fifteen years old he went to St. John's College, Cambridge: after five years close attention to the Mathematics, Astronomy, &c. he set out for the Continent; returning the next year, he was

printed at London in 1570, mentions Algebra as a mystery scarcely heard of by the studious in Mathematics here. In 1558, Peletarius published at Paris a very ingenious and masterly composition, entitled *Jacobi Peletarii Cenomani, de occulta parte Numerorum, quam Algebram vocant, Lib. duo;* in which he ably treats of all the parts of the subject then known, excepting Cubic Equations, and teaches some curious properties of square and cube numbers, with the method of constructing a table of each, by addition only; how to reduce trinomial surds to rational quantities, and even to simple ones, by means of certain compound multipliers; and that the root of an Equation is one of the divisors of the known or absolute term.

The *Stratioticos* of Mr. Thomas Digges[1], containing a

elected fellow of Trinity College. His great application to Astronomy, together with some ingenious mechanical inventions with which he occasionally amused himself, gave rise to a suspicion that he was a conjurer, and he was obliged in consequence to quit the country. He went to the University of Louvain, thence to the College of Rheims, where he read lectures upon Euclid; in 1551 he returned to England, and had the rectory of Upton upon Severn: afterwards, in consequence of a correspondence he had with Elizabeth, he was accused of practising enchantment against the life of Queen Mary her sister, and suffered a tedious confinement. On the accession of Queen Elizabeth, he was introduced to her, and (agreeably to the superstitious customs of that period) consulted respecting a propitious day for the coronation. She employed him afterwards in making geographical descriptions and maps of the countries to which England might have any claim; in this he acquitted himself with credit, as he did in his labours respecting the reformation of the calendar. In 1581, meeting with Edward Kelly, a credulous alchymist, our eccentric author and he performed together divers imaginary incantations, and held a pretended intercourse with angels and spirits. Our two conjurers asserted, that they were in possession of the secret of transmuting the baser metals into gold; and meeting with one Albert Laski, a Polish Nobleman, as credulous and ridiculous as themselves, they all three set out together for the Continent: here they imposed upon such of the rich and affluent as were silly enough to believe them, and lived on the profits of their trade in great affluence. Some disputes arising, Dee returned to England, and was graciously received by the Queen, who in 1595 made him Warden of Manchester College. He died at Mortlake in 1608, leaving some valuable works behind him.

[1] Thomas Digges flourished in the reign of Elizabeth, but the times of his

tract on Algebra, was published in 1579; the author was Muster Master General of the forces, and a man of great credit, so that, in all probability his work gave a fresh impulse to the study of Algebra in England, where it had for many years been on the decline; at least we know, that it now began to be studied and cultivated with more ardour than heretofore.

The next writers of note were Ramus[m], Bombelli[n], and Clavius[o], whose excellence consisted in their com-

birth and death are not known. Having studied for some time at Oxford, in consequence of what he acquired there, and the subsequent instructions of his learned father, he became one of the best mathematicians of that time; we have several very useful mathematical works of his still in print, and he left several others in manuscript.

[m] Peter Ramus was born at Vermandois in Picardy, in 1515. A thirst for learning, accompanied with extreme poverty, (for although he was of a good family, a series of misfortunes had made him poor,) urged him to become a servant in the College of Navarre, where he spent the day in performing the duties of his office, and most part of the night in study; by this means he acquired Classical learning, Rhetoric, Mathematics, and a knowledge of Philosophy. On taking his Master of Arts' degree, he defended a thesis, which went to overturn the whole doctrine of Aristotle, which at that time prevailed in the schools; this of course gave offence, but the force of his arguments proved an over-match for those of his adversaries. He was a great orator, sober, temperate, and chaste. He lay upon straw, rose early, and studied hard; being a Protestant, he endeavoured to shelter himself by concealment, during the dreadful massacre of St. Bartholomew in 1572; but was unfortunately discovered, dragged out, and murdered with circumstances of inhuman barbarity too shocking to relate.

The mathematical works of Ramus were enlarged, improved, and published in 2 volumes 4to. by Schoner; his Geometry, which is chiefly practical, was translated into English by Bedwell, and published at London, 4to. 1636.

[n] Raphael Bombelli was a mathematician of Italy; his Algebra was written in 1572, and published seven years after at Bologna.

[o] Christopher Clavius was born at Bamberg in Germany in 1537: he was a Jesuit, and cultivated mathematical learning, which he laboured at for more than 50 years; he assisted under Pope Gregory XIII. in the reformation of the calendar, which he afterwards defended against Scaliger, Vieta, and others. His stile of writing is considered as heavy, and his works are mostly elementary, forming a complete system, in 5 large volumes folio. He died at Rome in 1612.

ments on, and explanations of, preceding authors, rather than in any material improvements of their own.

Simon Stevinus [p], of Bruges, mathematician to Prince Maurice of Nassau, and inspector of the Dykes in Holland, was the author of several useful treatises in various branches of the Mathematics; among which, one is on Arithmetic, published in 1585, and another shortly after on Algebra: the latter, with which we are now particularly concerned, is an original and ingenious work, containing a variety of improvements. He here introduces the use of fractional exponents, whereby all sorts of roots are denoted, like powers by numeral indices: the notation of coefficients by including fractions, radicals, and numbers of every description, he greatly improved, and gave a general method of resolving all Equations by numbers; he likewise first applied the word *nomial* to all compound algebraic expressions, as *binomial*, *trinomial*, *quadrinomial*, *multinomial*, &c. the former of which, as has been observed, was first employed by Recorde.

As early as 1575, Gulielmus Xylander published at Basil a Latin translation of the six first books, and part of the seventh, of the Arithmetics of Diophantus [q], ac-

[p] Simon Stevin was very skilful in both Mathematics and Mechanics: he is said to have invented the sailing chariots sometimes used in Holland. His works on various branches of the Mathematics were written in Dutch, and translated into Latin by Snellius, making 2 volumes folio; but the best edition is that in French, with additions and notes, by Albert Girard, printed at Leyden in 1634. He died in 1633.

[q] Diophantus is supposed by some to have flourished before Christ, others place him in the second century after, and others again in the fourth, after; he lived at Alexandria, and is reputed to be the same who wrote the *Canon Astronomicus*, which was honoured with a commentary by Hypatia, the celebrated and unfortunate daughter of Theon, whom she succeeded as President of the Alexandrian school.

It does not appear in what manner the six books of Diophantus were recovered; Regiomontanus mentions them as being deposited in the Vatican Library in his time, and Bombelli proposed to translate them, but did not. That Diophantus was not the inventor of Algebra, as some have supposed, nor of the

companied with the Greek Scholia of Maximus Planudes, and notes. This work, which had been lost for many ages, consisted originally of thirteen books, but those only which we have mentioned have been published; the remainder, it is supposed, are irrecoverably lost. The Problems of Diophantus are of the kind called *indeterminate*, relating to square and cube numbers, right angled triangles, &c. and "so exceedingly curious and abstruse, that nothing less than the most refined Algebra, applied with the utmost skill and judgment, can surmount the difficulties which attend them."

The method of Diophantus was found to differ very widely from that of the Arabs, which had hitherto been followed, and it furnished succeeding algebraists with ample means of extending and perfecting the science. The first who availed himself of this advantage, was Franciscus Vieta[r], Master of the Requests to Mar-

Analysis of indeterminate Problems, as M. Bossut asserts, appears from the nature of his problems, and the consummate skill their investigation requires, which indicate such a maturity in the science as would require ages to produce. Xylander was a native of Augsburg, and became Professor of Greek at Heidelberg; he translated Diophantus, Plato, Dion Cassius, Strabo, and Marcus Antoninus; "he was very learned, and very poor, and laboured rather for bread than for fame, which helps to account for the numerous errors found in his writings." Born 1532, died 1576. *Melchior Adam, in Vitis Philosophorum. Bayle.* Maximus Planudes, the Scholiast, was a Monk of the Greek Church, and flourished at Constantinople in the fourth century; he was author of a collection of Epigrams, and of some Fables, which he ascribed to Esop, whose life he wrote; but the account he has given is said to be full of anachronisms, absurdities, and lies.

[r] Vieta was born in 1540 at Fontenai-le-Comté, in Lower Poitou; he excelled in various branches of learning, especially the Mathematics, scarcely any part of which is not indebted to his original and masterly genius for great and valuable improvements: to particularize them would require a volume; Algebra, Geometry, Trigonometry, Astronomy, are particularly indebted to him. By means of his angular sections, he resolved the famous problem of 45 dimensions, proposed by Adrian Romanus to all the world. His skill as a decipherer proved a great benefit to his country, during the troubles of the League, by disconcerting the councils of the Spanish court for more than two years.

garet, Queen to Henry IV. of France: his affection for the Mathematics was so great, that it is said he frequently passed three whole days and nights in study, without food or sleep; by him the Greek and Arabian methods were judiciously blended, and more improvements introduced than any former writer could claim. He was the first who introduced the general use of letters into Algebra, denoting the known quantities in a problem by the consonants, and unknown ones by vowels. He improved the method of reducing cubic and other Equations; shewed how to change the roots in a given proportion; how to raise cubic and biquadratic Equations from quadratics, by squaring and otherwise multiplying certain parts of the latter; he made various observations on the limits of the roots of Equations, and stated the general relation between the roots and coefficients, when the signs of the terms are alternately + and −, and none of the terms wanting; he gave the construction of certain Equations, and exhibited their roots by means of angular sections, a method which had been before adverted to by Bombelli. He introduced the vinculum, and the names *coefficient, affirmative, negative, pure, affected, unciæ*, &c. and was, I believe, the first who extracted the roots of Equations by approximation. The next whose writings deserve to be mentioned, is Albert Girard [a], an ingenious Dutch mathematician; his *Inven-*

The dark and crooked policy of the ambitious Philip induced him to support the Duke of Guise, and their correspondence was carried on by means of a cipher, consisting of above 500 different characters; this falling into the hands of the King's party, was successfully interpreted by Vieta, which was considered as so difficult a task, that many ascribed it to magic. Vieta died at Paris in 1603, and his works were collceted and published at Leyden in 1646, by Schooten, besides a large folio volume mentioned by Dr. Hutton as published at Paris in 1573. *Introduction to Hutton's Mathematical Tables, 2nd edit. p. 4. &c.*

[a] The time of his birth is not known; he died about the year 1633, leaving behind him the character of an ingenious and useful mathematician.

tion Nouvelle en l'Algebre, published at Amsterdam in 1629, presents us with several useful discoveries. It appears that he was the first who understood the use of negative roots in the solution of geometrical problems, and the general doctrine of the formation of the coefficients of powers from the sums of the roots, their products, &c. or gave rules for summing the powers of the roots of an Equation. He was the first who treated of imaginary roots, and understood that every Equation might have as many roots, real and imaginary, and no more, as there are units in the index of the highest power; he introduced the *parenthesis* as a convenient substitute, in many cases, for the vinculum: he was the first who employed the table of sines in the solution of the irreducible case of Cubics; and who distinguished negative quantities by the ridiculous appellation of *quantities less than nothing*; a name which has given some very able teachers much unnecessary trouble to explain.

One of the most scientific men that this nation ever produced, was Thomas Harriot[a], who, for his great and valuable improvements, is justly considered as the father of modern Algebra; his work on the subject, entitled *Artis Analyticæ Praxis, &c.* which was published in 1631, after his death, by his friend Warner, will ever remain a monument of his superior skill as an

[a] Thomas Harriot was born at Oxford in 1560: he was a commoner at St. Mary's Hall, where he took his Bachelor's degree in 1579. His uncommon proficiency in the Mathemathics attracted the favourable attention of Sir Walter Raleigh, who took him into his family in the character of a preceptor, allowing him a handsome pension. In 1584 he went to Virginia with Sir Walter's new colony, where he was employed in surveying and making maps of the country. He died in 1621 from an ulcer formed in his lip, in consequence, it is said, of holding the brass mathematical instruments, while working, in his mouth. Harriot was undoubtedly the most eminent algebraist of his time.

analyst: this work is a specimen of great and original genius, and supplies the first instance of the form of Algebra at present in use. He here, first of any, shewed the universal generation of all affected Equations by the continual multiplication of simple ones; "thereby exhibiting to the eye all the circumstances of the nature, mystery, and number of roots, with the composition and relations of the coefficients," from whence many important properties have since been deduced; "he greatly improved the numeral Exegesis, or the extraction of the roots of all Equations, by clear and explicit rules and methods, drawn from the foregoing generation of affected Equations;" and to him we are indebted for many original and elegant solutions of quadratic, cubic, and biquadratic Equations, which have since been discovered among his manuscripts; he was likewise the inventor of the signs $>$ *greater than*, and $<$ *less than*.

The Reverend William Oughtred[a], rector of Aldbury in Surry, and one of the first mathematicians of the time, published in 1631 his *Clavis Mathematicæ*, which was originally written for the use of Lord William Howard, his pupil. "His style and manner were very concise, obscure, and dry; and his rules and precepts so

[a] William Oughtred was born at Eton in 1573, and educated at the school there; in 1592 he went to King's College, Cambridge, where he spent 12 years, and became a fellow; he was presented to the living of Aldbury in 1603, and about 1628 was appointed Tutor to Lord William Howard, by Lord Arundel, his father. The chief mathematicians of the age were indebted to Oughtred for much of their skill, his house being open to all who came for instruction: after a life of temperance, exercise, study, and zeal in performing effectually the duties of his sacred function, he died, it is said, of a sudden extasy of joy, on hearing that the Parliament had passed a vote for the restoration of Charles II. Besides the works on Arithmetic, Algebra, Geometry, Trigonometry, &c. published in his life time, such of the manuscripts which he left, and which were found proper for the press, were published in 1676 at Oxford, in 8vo. under the title of *Opuscula Mathematica hactenus Inedita*, &c.

involved in symbols and abbreviations, as rendered his mathematical works difficult to be understood;" nevertheless, his writings were considered as valuable, and are still held in great esteem among the learned. Besides some characters not at present in use, he introduced the sign × for multiplication, :: for proportion, ∺ for continued proportion, ⊐ for greater, and ⊏ for less, with the method of multiplying or dividing by the component parts of a number, instead of the number itself; of multiplying and dividing decimals by the contracted method usually taught in our schools at present, with other neat and convenient abbreviations.

The application of Algebra to geometrical lines and curves, began about this period to exercise the skill of the learned. Fermat, a learned and ingenious French mathematician, was the first who successfully cultivated this branch; but it was Des Cartes [x], who incorporated

[x] Réné Des Cartes was descended from an ancient noble family in Touraine, and born in 1596. At eight years old he was placed under the tuition of Father Charlet, at the Jesuits' College of La Fléche; conceiving a dislike to philosophy, he quitted the College in 1612, proposing to himself a military life: for this purpose he speedily acquired the necessary accomplishments, but a weak constitution rendering him unfit for the duties of a martial profession, he went to Paris, where, by the advice of Father Mersenne, and others of his learned acquaintance, he was prevailed on to resume his studies; after two years he returned to the army, and was successively a volunteer in the service of the Prince of Orange, and the Duke of Bavaria. After this he travelled for improvement, and was indefatigable in the study of almost every branch of science; "he extended," says Voltaire, "the limits of Geometry as far beyond the place where he found them, as Newton did after him."——"He employed this geometrical and inventive genius to Dioptries, which, when treated by him, became a new art." Voltaire acknowledges, that the rest of Des Cartes' works contain innumerable errors: indeed, his system of the universe, which is called *The Cartesian Philosophy*, and which prevailed until Sir Isaac Newton's system supplanted it, depends solely on hypothesis; and the theory of vortices, on which the Cartesian system is founded, is absolutely false. See *Dr. Keill's Examination of Burnet's Theory, &c.* Des Cartes died at Stockholm in 1650.

and improved the theories and observations of preceding writers, and first gave the doctrine a form and consistence. The Geometry of Des Cartes was published in 1637: it is properly neither Algebra purely, nor Geometry, but the application of the one to the other; nevertheless it contains improvements in both. The principal of those which relate to the present subject, are the following; viz. the geometrical construction of the higher orders of Equations, whereby the nature and properties of their roots, positive, negative, and impossible, are clearly elucidated. The rule for resolving biquadratic Equations, by means of a cubic and two quadratics, which usually goes by his name; and he is the first who denoted the unknown quantities in an Equation, by the final letters x, y, z, w, &c. and the known ones by the initials a, b, c, d, &c. according to the present mode of practice. But while we do justice to the superior talents of this distinguished philosopher, it must be acknowledged, that several of his methods and observations, which at the time of their publication were considered as new, were afterwards traced to the writings of Harriot, the English Algebraist [y].

The Geometry of Des Cartes, now called the *New Geometry*, was soon cultivated with ardour and success. Francis Van Schooten, a Dutch mathematician, translated it out of French into Latin, adding a commentary of his

[y] M. Roberval, a member of the Academy of Sciences, and Professor of the Mathematics at the College Royal, was once (shortly after the above work appeared) extolling the ingenuity of Des Cartes, for his contrivance in placing all the terms of an Equation on one side, and making the whole equal to nothing; upon which Sir Charles Cavendish, who was present, hinted that the praise was due—not to Des Cartes, but to Harriot: a few days after Sir Charles produced Harriot's Algebra, and Roberval, after he had examined it, exclaimed, *Oui, il l'a vu! il l'a vu!* "Yes, he has seen it! he has seen it!" See *Dr. Wallis's Algebra*, p. 198.

own, and notes by M. De Beaune, 1649. Schooten's *Principles of Universal Mathematics* appeared in 1651, and *Exercitationes Mathematicæ*, six years after; in both which are much excellent matter, and a variety of curious analytical pieces. *The Method of Indivisibles* of Cavalerius [z], was published in 1635, and proved a new æra in analytics, from whence arose new modes of computation. Our learned countryman, Dr. John Wallis [a], was the author of

[z] Bonaventura Cavalieri was a native of Milan, a Friar of the order of the Jesuati of St. Jerome, a disciple of Galileo, the friend of Torricellius, and Professor of Mathematics at Bologna, where he died in 1647, leaving behind him several learned treatises on Geometry, Trigonometry, Logarithms, &c. besides the above-mentioned work on Indivisibles.

[a] Dr. John Wallis was born at Ashford in Kent, in 1616; after acquiring a tolerable proficiency in the Latin, Greek, Hebrew, and French languages, with the rudiments of Logic, Music, &c. he went to Emanuel College, Cambridge: he took orders, and obtained a fellowship of Queen's College; after this he was Chaplain, first to Sir Richard Darley, and then to Lady Vere. In 1642 he was appointed Savilian Professor of Geometry at Oxford, a situation which he filled with great ability; five years after he took the degree of Doctor in Divinity, and the next year his controversy with Mr. Hobbes commenced; this and his dispute with Mr. Stubbe lasted more than three years, during which several pamphlets were written on both sides, and the Doctor displayed a degree of spirit, moderation, and address, highly creditable to himself. In 1658 he was chosen *Custos Archivorum* of the University; King Charles II. respected him both for his talents, and for his attachment to himself and to his unfortunate father; and in consequence our author was, on the restoration, confirmed in all the places he held, and appointed one of the Chaplains in Ordinary, and likewise to assist in revising the book of Common Prayer. He was a very industrious and useful member of the Royal Society, and kept up a constant literary correspondence with most of the principal learned men of the time. Dr. Wallis died at Oxford in 1703, leaving behind him one son and two daughters; the Rev. Henry Peach, B.D. the present worthy Rector of Cheam, to whose kindness the writer acknowledges himself, with gratitude, under great and repeated obligations, is grandson of one of the latter. The publications of Dr. Wallis comprehend a great variety of subjects, and are for the most part filled with learned and useful matter; and he has contributed to nearly all the first 25 volumes of the Philosophical Transactions. His mathematical works, which had appeared separately, were published together by the University in 1699.

many useful works on the Mathematics, especially his *History and Practice of Algebra;* but that which gained him the greatest credit was his *Arithmetic of Infinites,* published in 1655, being a new method of reasoning on quantities, and a great improvement on the Indivisibles of Cavalerius.

This subject, as treated by Dr. Wallis, was evidently the ground-work of Sir Isaac Newton's most valuable discoveries in Analysis, which took place shortly after; such as *the Binomial Theorem; the Method of Fluxions and Infinite Series; the Quadrature, Rectification, &c. of Curves; the Investigation of the Roots of Equations, both numeral and literal, by means of Infinite Series; the Reversion of Series,* &c. &c. all of which were written between the years 1665 and 1727. Every lover of truth must revere the name of Newton; and we ought to be thankful to Providence for the uncommon endowments of his mind, whereby the fetters of prejudice were broken, and mankind released from the tyranny of error and hypothesis, so nearly connected with the interests of religion.

In 1668 Mr. Thomas Brancker [b] published *Rhonius's Algebra,* with the additions of Dr. John Pell [c]. The

[b] Thomas Brancker was born in 1636: he was admitted Batler of Exeter College, Oxford, in 1652; three years after, he took his Bachelor's degree, and was elected a fellow; in 1658 he took the degree of Master of Arts, and became a preacher: some time after he was minister of Whitegate, his skill in the sciences recommended him to the favour of Lord Brerewood, who gave him the living of Tilston. He was afterwards chosen Master of Macclesfield School, where he died in 1676; a work of his in Latin, on the Sphere, appeared in 1662.

[c] Dr. John Pell was born in 1610, and at thirteen years of age was sent to Trinity College, Cambridge. His eminent mathematical knowledge procured him in 1643 the Professorship of Mathematics at Amsterdam; thence he removed in 1646 to the New College at Breda, where he held the same office with an increased salary; about six years after he returned to England, and was employed by Cromwell in a diplomatic capacity, in which he acquitted himself with credit; after this, on receiving orders, he became Rector of Fobbing in Essex, in 1661, to which was added, twelve years after, the Rectory of Laingdon, in the same

Doctor here first introduces the method of registering the steps of an Equation, by characters placed for reference in the margin: he likewise invented the signs ÷ for division, $\odot$ for involution, and *ω* for evolution.

Dr. Barrow's[d] *Method of Tangents* was published in

county. About this time he took his degree of Doctor in Divinity, and was made Chaplain to Archbishop Sheldon; but his thoughts being too much employed in the cultivation of the sciences, to allow him time for the management of his pecuniary concerns, he was so cheated by his tenants and dependants, that he wanted money to purchase the common necessaries, and he found difficulty to procure even pens, ink, and paper; this brought him to the King's Bench Prison, and to a state of dependance after his release: he died at the house of a friend, in 1685; and his funeral expences were defrayed jointly by Mr. Sharp, Rector of St. Giles's, and Dr. Busby. Dr. Pell published a number of mathematical books, and left behind him a large collection of papers and letters on mathematical and other subjects, which are in the hands of the Royal Society; others he left at the seat of Lord Brereton, at Brereton, in Cheshire.

[d] Isaac Barrow was born at London in 1630. He first went to the Charter House, where his behaviour was so bad, that little hopes of him were entertained by his friends; and his father has been heard to say, that, "if it pleased God to take either of his children, he hoped it would be Isaac." He was entered at Trinity College, Cambridge, in 1645, and now began to apply himself in earnest to learning. Mathematics, Natural Philosophy, Chronology, Physic, Anatomy, Chemistry, Botany, and Divinity, were the objects of his unremitting attention. Having studied these with great success, and meeting with a disappointment at the University, he quitted it, and in 1655 set out on his travels on the Continent; he returned about four years after, and having been ordained, was chosen in 1660 Greek Professor at Cambridge; two years after he was made Professor of Geometry at Gresham College, where he discharged the duties both of the geometrical and astronomical departments. The Royal Society elected him a Fellow in 1663, and the same year he was appointed the first Lucasian Professor of Mathematics at Cambridge. At length, being determined to direct his studies wholly to divinity, he resigned the mathematical chair (which he had filled with much credit) to his friend, Mr. Isaac Newton. In 1670 he was created D. D. by Mandate, and in 1672 King Charles I. appointed him Master of Trinity College by patent, observing, that "he had given it to the best scholar in England." Dr. Barrow died in 1677, and was buried in Westminster Abbey: his numerous writings on a great variety of subjects, do an honour to his country. "He was unrivalled," says Mr. Grainger, "in mathematical learning, especially in the sublime Geometry, in which he has been excelled only by his successor, Newton; he at length gave himself up entirely to divinity, and particularly to the most useful part of it, that which has a tendency to make men wiser and better."

1669; and about 1677 Leibnitz[e] discovered his *Methodus Differentialis,* being either a variation of Newton's Fluxions, or else an extension of Barrow's Method of Tangents above-mentioned.

In 1693 appeared M. Ozanam's[f] *Course of Mathematics,* in 5 volumes octavo, containing a treatise on Algebra; nine years after he sent forth a work exclusively on Algebra, in which the problems of Diophantus are skilfully handled. He was the author of several other mathematical works, among which may be mentioned his *Mathematical and Philosophical Recreations,* translated and published by Dr. Hutton, which afford both amusement and instruction. A book on the same subject had been published many years before by Bachet de Meziriac, under the title of *Problêmes plaisans et délectables sur les Nombres*[g].

The invention of Fluxions opened the way for new discoveries and improvements. Mathematicians in every

[e] Godfrey William Leibnitz was born at Leipsigc in 1646, where, and at Jena, he received his mathematical education; his writings, which are numerous and valuable, were published separately, or among the memoirs of different academies: as a mathematician, he is chiefly remarkable for laying claim to the invention of Fluxions, in opposition to Sir Isaac Newton, and for his system of Philosophy, intended as an amendment of the Cartesian, in opposition to the Newtonian Philosophy. In his system are retained the subtile matter, the vortices, and the universal plenum of Des Cartes; and the universe is considered as a vast machine, which by the laws of mechanism will, by absolute necessity, continue in motion for ever in the most perfect state. He died in 1716.

[f] Jaques Ozanam was born at Boligneux in France in 1680: he was a respectable teacher of the Mathematics, first at Lyons, then at Paris, and published a great number of very useful books on various branches of the Mathematics, with some pieces in the *Journal des Scavans,* the *Memoirs of the Academy of Sciences,* the *Mémoires de Trevoux,* &c. Ozanam died in 1717.

[g] The *Récréation Mathématique* of H. Van Etten is a well known work, and principally remarkable for its absurdities; Mydorgius undertook to correct them, but was not very successful, as appears by the later editions of the book, which are of little value.

The Greek Anthology is the earliest example we have of this kind of composition.

part of Europe, aided by the light which this sublime discovery had thrown on the new Geometry of Des Cartes, were intent on extending the bounds of the science, and their labours were attended with astonishing success. *The Method of Increments,* which naturally arises out of the doctrine of Fluxions, was discovered by Dr. Brook Taylor, who published, in 1715, his *Methodus Incrementorum;* this ingenious and learned work appearing too difficult for the generality of readers, Mr. Emerson undertook to render the subject more easily attainable; and succeeded probably beyond his own expectation, in a work which appeared in 1763.

The calculation of the probabilities in the theory of games of chance, owes its origin to M. Pascal [h], resulting from one of the numerous applications of the doctrine of his celebrated arithmetical triangle. Some of the greatest mathematicians of the time directed their attention to the subject, but Huygens [i] was the first who wrote expressly on it, in his treatise *De Ratiociniis in*

[h] Blaise Pascal was born at Clermont, in Auvergne, in 1623, and was one of the greatest and best writers France has ever produced. He was a very eminent mathematician: "his writings," says Voltaire, "may be considered as models of eloquence and humour;" his Provincial Letters, which do equal honour to his head and heart, have been published in every language and country in Europe. M. Bayle says, that a hundred volumes of sermons are not of so much avail as the history of this great man; he calls him *a paragon in the human species;* and adds, "when we consider his character, we are almost inclined to doubt whether he was born of a woman." Pascal died at Paris in 1662, and his works were collected by L'Abbé Bossu, and published both at Paris, and at the Hague, in 1779, in 5 volumes octavo.

[i] Christopher Huygens was born at the Hague in 1629: he discovered an early love for the Mathematics, and studied them with uncommon ardour under the celebrated Schooten at Leyden: in 1655 he went to France, where he had the degree of *Doctor of Laws* conferred on him; and five years after, visiting London, he was made a Fellow of the Royal Society. The talents of this great man were employed, not merely in promoting the interests of speculative science, but in reducing his knowledge to practical uses for the benefit of man. He was the first who discovered Saturn's ring, and also the third satellite of that planet; he first applied the pendulum to clocks, and equalized its

Ludo Aleæ. The posthumous work of James Bernoulli [k], entitled *De Arte Conjectandi,* is well known; Nicholas Bernoulli, the nephew of that excellent man, made an important application of the principles contained in his book to the probabilities of the continuance of human life. *The Analysis of Games of Chance,* by Remond de Montmort, appeared in 1711, is a work of great merit, and has for its object the subjecting of probabilities of every kind to calculation.

The importance of the theory which had employed the talents of these illustrious foreigners, soon appeared from the useful applications made of it, especially in England, to the uses and purposes of life; such as the

vibrations by means of the cycloid; and perfected the telescope then in use. In 1681, Huygens was admitted a member of the Academy of Sciences at Paris, which was the last public mark of honour he received; for his intense application having impaired his health, he retired as often as he could into the country. He died at the Hague in 1695, and his principal works being collected, two editions of them were published in 4to, under the inspection of Professor S'Gravesande, the first at Leyden, 1682, the other at Amsterdam, 1728.

[k] James Bernoulli was born at Basil in 1654, and died in 1705: he was Professor of Mathematics at Basil, and member of the Academies of Sciences of Paris and Berlin. John Bernoulli, brother of the above, and equally famous, was born at Basil in 1667, where he obtained the degree of Doctor of Physic; he was first, Professor of Mathematics at Groningen, but on his brother's death, was appointed to succeed him in the mathematical chair at Basil; he was member of most of the Academies of Europe, and both he and his brother cultivated the new analysis with the greatest application and success. He died in 1748. Daniel Bernoulli, the son of John, was born at Groningen in 1700; he became Professor of Physic and Philosophy at Basil, and on his father's death, succeeded him in the Academy of Sciences. It is remarkable that this Academy, from its first institution in 1699, had never been without a Bernoulli; after a long, useful, and honourable life, Daniel Bernoulli died, much lamented, in the year 1782. John Bernoulli, grandson of the above John, is well known as an industrious and skilful astronomer at Berlin. His brother, James Bernoulli, was born at Basil in 1759; he read lectures on Philosophy in the University of Basil for his uncle Daniel: afterwards he was secretary to Count Brenner, the Emperor's envoy at Venice, and in 1786 became a member of the Academy of Sciences at Petersburg, where he died in 1789. Of Nicholas Bernoulli I have not been able to obtain any satisfactory account

valuation of annuities, assurances, reversions, &c. Mr. Abraham Demoivre[l] appears to have been the first in this country, who distinctly explained the doctrine and mode of application, in his *Doctrine of Chances, &c.* 1718, and his *Annuities on Lives*, 1724: indeed he was particularly famous for calculations of this kind, spending most of his time upon them. Mr. Demoivre was ably followed by Mr. Thomas Simpson[m], whose two treatises *On the*

[l] Abraham Demoivre was born at Vitri in Champagne, in 1667; he was an eminent teacher of the Mathematics at London, where he (being a Protestant) had been driven by the revocation of the edict of Nantz; he was a Fellow of the Royal Society, and member of the Academies of Sciences at Paris and Berlin, and was, in consequence of his consummate abilities as a mathematician, appointed by the Royal Society to determine the dispute between Newton and Leibnitz, concerning their respective rights to the invention of Fluxions. He died at London in 1754.

[m] Thomas Simpson was born at Market Bosworth in Leicestershire, in 1710: he was bred a weaver, but gave early indications of a thirst for knowledge of a superior kind; this not being agreeable to the views of his father, a separation soon took place, and young Simpson went to reside at Nuneaton. Here, by means of the instructions given him by an itinerant pedlar and fortune-teller, during his periodical visits, and our author's own endeavours, he acquired considerable acquaintance with the principles of Arithmetic and Algebra, and likewise sufficient skill in the art of conjuring to enable him, in the technical phrase, to answer all fair questions about futurity. We now find him working at his trade all day, teaching a school in the evening, and occasionally telling fortunes; this continued till a curious circumstance, the account of which is too long to relate, obliged him to quit both his home, and the trade of astrology: he then went to Derby, where he worked at his trade, and kept an evening school; but his emoluments being insufficient for the support of himself and family, he was induced to repair to London about 1736; he settled in Spitalfields, working at his trade of weaving, and occasionally teaching the Mathematics. He now became better known, and the number of his pupils increasing, he was encouraged to make proposals for publishing his work on Fluxions by subscription, which proved the precursor of many other learned and valuable works; and he now, for the first time, began to discover that the pretended art of astrology is nothing more than a system of fallacy and deceit. In 1743, he was appointed Professor of Mathematics at the Royal Military Academy, Woolwich; and two years after was elected Fellow of the Royal Society: he died in 1761, at Bosworth. A more complete and circumstantial account of Mr. Simpson is to be found at the beginning of his Select Exercises.

Nature and Laws of Chance, 1740, and *The Doctrine of Annuities and Reversions*, 1743, are held in great esteem. Mr. James Dodson has furnished a numerous collection of examples, to exercise the rules and theorems laid down by preceding writers, in the second and third volumes of his *Mathematical Repository*, published between the years 1747 and 1756. The merits of Dr. Price [*], as a writer on this subject, are well known to the public. His *Observations on Reversionary Payments, &c.* appeared in 1770, as did Mr. Morgan's *Doctrine of Annuities and Insurances, &c.* in 1779; the latter work being furnished with an Introduction, and an Essay on the present state of population in England and Wales, both by Dr. Price.

In 1717, Mr. James Sterling published *Lineæ tertii ordinis*, and in 1730, *Methodus Differentialis*; both of which abound in improvements in the higher branches of Algebra.

In 1740, Professor Saunderson [°] published his *Elements of Algebra*, which work has been considered as the most complete and comprehensive system in the English lan-

[*] The Rev. Dr. Richard Price, an eminent Dissenting minister, was born in 1723; his distinguished abilities as a calculator are well known. "He was," says his biographer, "the friend of man, and the most intrepid assertor of his rights; and while his genius, and his no less abstruse than valuable labours in calculation, rank him with the first philosophers of every age, his political counsels and writings place him among the most distinguished patriots and benefactors of nations." He died in 1791.

[°] Nicholas Saunderson was born at Thurlston in Yorkshire, in 1682, and lost his eye-sight by the small-pox when he was but a year old; notwithstanding which he became a very eminent mathematical and classical scholar. In 1707 he went to Cambridge, where he taught the Mathematics, and gave lectures on Newton's Principia; four years after he was appointed Lucasian Professor of the Mathematics. In 1728, King George II. visited the University, on which occasion our author was created LL.D. a degree which perhaps was never more worthily bestowed. Dr. Saunderson wrote with great clearness on almost every branch of the Mathematics. He died in 1739.

guage. Maclaurin's[p] Treatise of Algebra appeared shortly after his death, which happened in 1748: it is an excellent work, and has an Appendix on the application of Algebra to Curve Lines, in which the subject is very ably treated. The mathematical works of Mr. Thomas Simpson were published between the years 1740 and 1753; and are no less remarkable for their usefulness, than for the striking display of originality which they exhibit; especially those parts which relate to our present subject. Mr. William Emerson[q] published a complete course of the Mathematics, by single volumes, in 1763, and the succeeding years. His *Algebra* was first printed in 1764, and evinces incontestable marks of vigorous genius, accompanied with a rude and uncultivated taste. In 1764, came out Mr. Landen's[r] *Residual Analysis*, a new branch in the

[p] Colin Maclaurin was born at Kilmoddan, in Scotland, in 1698; received his education at the University of Glasgow, and took his Master of Arts' degree in his fifteenth year: in 1717 he became Professor of the Mathematics at the Marischal College of Aberdeen, and two years after a Fellow of the Royal Society. In 1725 he was elected Professor of the Mathematics at the University of Edinburgh, principally through the interest of Sir Isaac Newton, where he acquitted himself to the satisfaction of all. He was remarkably well skilled in the new Geometry, and in whatever was connected with the recent discoveries of Newton, Leibnitz, the Bernoulli's, Wallis, Cavalerius, and others; and engaged successfully in the solution of most of the great problems of the day: he was a great man, and as good as he was great. He died in 1746.

[q] William Emerson was born at Hurworth in Durham, in 1701, and received the whole of his education from his father, who was a schoolmaster, and the Curate of Hurworth, who lodged at his father's house; he once took pupils, but want of sufficient patience, and of a happy method of teaching, soon induced him to relinquish that plan. I have been informed, that he was a great adept at thatching, that he often hired himself out to that employ, and preferred it to teaching. He was a very ingenious mathematician, but excelled in mechanics, in which it was his constant practice to prove his conclusions by experiments made with machines of his own construction. The eccentricity of his character was obvious from his complete disregard of dress. He died at Hurworth in 1782.

[r] John Landen was born at Peakirk in Northamptonshire. He was a very respectable mathematician, and a contributor to the Ladies' Diary: in 1766 he

algebraic art, capable of resolving problems to which Fluxions are usually applied, and in a more natural and elegant manner; other mathematical works have since been published by this ingenious author.

The Elements of Algebra, by the celebrated Leonard Euler[*], were published in 1770, in the German language: in 1774 it was translated into French, and a translation of it into English has lately been announced by the booksellers; this work, especially the French translation of it, has been much commended. In 1789, Mr. James Glenie published his *Doctrine of Universal Comparison;* and in 1793, *The Antecedental Calculus*, or a geometrical method of reasoning, without any consideration of motion or velocity, applicable to every purpose to which Fluxions have been, or can be applied: with the geometrical principles of Increments, &c.

In 1798, Professor Vilant, of Edinburgh, published *The Elements of Mathematical Analysis, abridged; with Notes, demonstrative and explanatory; and a Synopsis of*

was elected Fellow of the Royal Society, and enriched its Transactions with some curious and valuable papers. His *Mathematical Lucubrations* were published in 1755; and in 1781, and the two following years, he gave three small tracts on Infinite Series. He died at Milton in 1790.

[*] Leonard Euler, one of the greatest mathematical geniuses the world ever produced, was born at Basil in 1707; at the University of which he received his education, and where he attracted the particular regard of the celebrated John Bernoulli, and became the intimate friend of his two sons, Nicholas and Daniel: to this circumstance are justly attributed his vast acquirements in mathematical and philosophical knowledge, to do justice to which an ample treatise would scarcely be sufficient. He took the degree of M. A. in 1723, and shortly after became joint Professor with Herman, and Daniel Bernoulli, at Petersburg. In 1730 he was promoted to the Professorship of Natural Philosophy, and three years after succeeded his friend, Daniel Bernoulli, in the mathematical chair: he was a member of the Academy of Sciences at Paris, and enriched the memoirs of that, and of several other Academies, with a vast number of invaluable papers; the catalogue only of his works occupies fifty pages. This great man died at Petersburg, in the year 1783, universally lamented.

Book V. of Euclid. This work is an abridgment of another work *not finished*, and although very negligently printed, it is calculated to make a useful text book for those who lecture on the subject: the rules, solutions, and demonstrations being given with great clearness; to which are added proper marginal references.

Mr. Wood's [t] *Elements of Algebra, for the use of Students in the University* of Cambridge, forming a part of the system of mathematical instruction, undertaken jointly by himself and Mr. Vince, appeared about the same time. This excellent work would be improved by a greater number and variety of examples.

Dr. Maskelyne's [u] method of resolving all the cases of cubic equations, by the tables of sines, tangents, and secants, will be found in Taylor's Logarithms: the improvements and discoveries of Dr. Waring [x] are contained in his *Meditationes Algebraicæ*, 1770; *Proprietates Algebraicarum Curvarum*, 1772; *Meditationes Analyticæ*, 1776, and in the Philosophical Transactions: those of the Rev. Mr. Vince [y], in the Philosophical Transactions, and in his *Fluxions*, and other works: and those of Dr.

[t] The Rev. James Wood, B. D. Fellow and Tutor of St. John's College, Cambridge.

[u] Neville Maskelyne, D. D. F. R. S. late Astronomer Royal at the Royal Observatory, Flamsted House, Greenwich: he died in the spring of 1811, and was succeeded by Mr. Pond, a gentleman whose abilities as an astronomer are well known.

[x] Edward Waring, M. D. the late Lucasian Professor of Mathematics at Cambridge, was the most profound analyst in Europe, as is clearly evinced in his *Miscellanea Analytica*, published in 1762; *Proprietates Algebraicarum Curvarum*, 1772; *Meditationes Analyticæ*, 1776; and in his various papers and tracts published in the Philosophical Transactions, and elsewhere. The profound skill and deep penetration displayed in many parts of the writings of this learned author, have excited the admiration of mathematicians in every part of Europe. He died in 1798.

[y] The Rev. Samuel Vince, M. A. F. R. S. the present Plumian Professor of Astronomy and Experimental Philosophy at the University of Cambridge.

Hutton [a], in his *Mathematical Tracts*, and other parts of his numerous and useful publications.

Mr. Bonnycastle's improvement of the binomial theorem, its use in the construction of logarithms, and his observations on the irreducible case of cubic equations, with an improved solution by the table of sines, are to be met with in Dr. Hutton's Mathematical Dictionary. We must not, in this enumeration, omit the name of Mr. Baron Maseres; a gentleman, whose extensive genius and unremitting industry have been long successfully employed in cultivating the mathematical sciences; to whom Algebra in particular owes much: and it is in consequence of his liberal encouragement and patronage, that the works of the most distinguished female analyst the world could ever boast, are made our own [a]. By the united abilities of these, and of a far greater number of distinguished writers, whose names we are obliged to omit, the science of Algebra has attained to its present state of perfection [b].

[a] Charles Hutton, LL. D. F. R. S. the late Professor of Mathematics at the Royal Military College, Woolwich: he was succeeded by Mr. John Bonnycastle.

[a] The analytical works of Madame Agnesi, translated by Mr. Colson, and published under the inspection of the Rev. John Hellings, B. D. and F. R. S.

[b] Among the smaller elementary books on Algebra, may be mentioned the Introductions of Fenning, Bonnycastle, the Rev. Mr. Joyce, and *An Easy Introduction to Algebra*, by the author of this work, published in 1799. Likewise, *Lectures on the Elements of Algebra*, by the Rev. B. Bridge, B. D. Fellow of St. Peter's College, Cambridge, and Professor of Mathematics in the East India College; an easy and useful work.

DEFINITIONS AND NOTATION.

1. Algebra, as we have observed, is the science which teaches general methods of performing computations, by means of letters, signs, and other general characters.

2. Quantities are expressed by the small letters of the alphabet[b], to which the figures 1, 2, 3, &c. are sometimes joined; this will be fully explained in its proper place.

3. The sign + is the mark for Addition; it is named *plus*; (more,) and is placed between two quantities, to shew that the quantity which follows the sign is to be added to the quantity going before it.

Thus, $a + b$ is read *a plus b*, and signifies that the quantity represented by b, is to be added to the quantity represented by a.

4. Quantities having the sign + prefixed, are named *positive* or *affirmative* quantities: if a quantity has no sign prefixed, it is affirmative, + being understood; and if a positive (or affirmative) quantity stands alone, or on the left of all the others connected with it, then the sign + is usually omitted; but this sign is never omitted in any other case.

5. The sign — is the mark for Subtraction; it is named *minus*, (less,) and signifies that the quantity which follows the sign, is to be subtracted from the quantity going before it, or from the other quantities concerned.

Thus, $a-b$ is read *a minus b*, and shews that the quantity represented by b, is to be subtracted from that represented by a; also $-c + a + b$ shews that c is to be subtracted from the sum of a and b.

6. Quantities having the sign — prefixed, are called *negative* quantities: this sign is always prefixed to negative quantities, and must not be omitted in any case: likewise every quantity is either positive or negative, and consequently must have either +, (expressed or understood,) or — belonging to it.

7. Two or more quantities are said to have *like signs*, when

---

[b] In addition to these, the capitals are frequently employed, as are the Greek letters, the signs of the ecliptic, and in general, the characters peculiar to any subject to which Algebra may be applied.

the signs are either all $+$, or all $-$; and they are said to have unlike signs, when some are $+$, and others $-$.

8. The sign $\times$ is the mark for Multiplication; it is named *into*, and signifies that the quantities between which it stands are to be multiplied together.

Thus, $a \times b$, is read *a into b*, and shews that the quantity represented by a is to be multiplied *into*, or by, the quantity represented by b.

9. A point[c] is sometimes used instead of the sign $\times$; thus $a \times b$ may be written $a.b$, also $3 \times x \times y \times z$ may be written $3.x.y.z$.

10. Both the sign and point are frequently omitted; thus $a \times b$, or $a.b$, are frequently written thus, ab; also $2 \times x \times y$ is written $2xy$; and when several letters are placed together like the letters of a word, their product is always understood; thus $abcd$ denotes $a \times b \times c \times d$: and the same is to be understood when a number is connected with the letters; thus $3 \times a \times b \times c$, is written $3abc$; but when two or more numbers are connected, either the sign $\times$ must be interposed between the numbers, (not the point,) or their product must be taken; thus, $3 \times 4 \times x \times y$, must be written $3 \times 4\,xy$, or $12xy$, and not $3.4.x.y$, or $34xy$.

11. When the product (or multiplication) of two or more letters, is denoted by placing the letters together like a word, it is indifferent in what order they are placed; thus ab and ba are the same; also abc is the same as acb, or cba, or cab, or bca, or bac; but it is usual, for the sake of method, to place the letters in alphabetical order: likewise when a number is connected with letters by the sign $\times$, it is indifferent in what order they stand; thus, $a \times 3 \times b$, and $a \times b \times 3$, and $b \times a \times 3$, and $b \times 3 \times a$, and $3 \times a \times b$, and $3 \times b \times a$, are the same; but when the sign $\times$ is not interposed, the number must always stand first, thus, $3ab$, or $3ba$, and never $a3b$, or $b3a$, or $ab3$, or $ba3$.

12. The sign $\div$ is the mark for Division; it is named *by*, and signifies that the quantity standing before the sign, is to be divided by the quantity which follows it.

---

[c] The use of the point to denote multiplication was introduced (as Dr. Hutton supposes) by M. Leibnitz.

Thus, $a \div b$ is read *a by b*, and denotes that the quantity represented by a, is to be divided by that represented by b.

12. Division is likewise denoted by placing the dividend above the divisor, with a small line between, like a fraction; thus $a \div b$ is written $\frac{a}{b}$.

It may be remarked, that the signs $+$, $-$, and $\div$ always belong to (or govern) the quantity which follows the sign; thus in the expression $a - b + c$ . . . b is the quantity to be subtracted, not a; and c is the quantity to be added, not b; also in the expression $a \div b$ . . . b is the divisor, and not a. But the sign of multiplication $\times$ equally affects both quantities between which it stands; thus $a \times b$, denotes the multiplication of a by b, or of b by a, either may be considered as the multiplier.

13. The sign $=$ is the mark for equality; it is named *equal* or *equals*, and denotes that the quantity or quantities on one side of the sign, are equal to the quantity or quantities on the other side.

Thus $a = b$, is read *a equals b*, and denotes that the quantity represented by a, is equal to that represented by b; also $a + b = c - d$, is read *a* plus *b equal c* minus *d*, and shews that the *sum* of a and b is equal to the *difference* of c and d.

14. The coefficient of a quantity is the number prefixed to it. Thus in the quantities $2a$, $3bd$, and $4xyz$, 2 is the *coefficient* of a, 3 is the *coefficient* of bd, and 4 is the *coefficient* of xyz; these numbers are sometimes called *numeral coefficients.*

In quantities consisting of two or more letters, placed together like a word, any one or more of the letters may be considered as the *coefficient* of the rest; thus in the quantity ab. . . a is the *coefficient* of b, and b of a; in the quantity xyz . . . x is the *coefficient* of yz, y of xz, and z of xy; in like manner yz is the *coefficient* of x, xz of y, and xy of z: these, in order to distinguish them from the numeral ones, are called *literal coefficients.*

Every algebraic quantity has a numeral coefficient, either expressed or understood; when 1 is the coefficient, it is seldom put down, and when a quantity has no coefficient, 1 is always understood to be its coefficient; when the coefficient is greater than 1, it is always expressed.

15. A simple quantity is that which is represented by one

number or letter, or by several letters standing together like a word, or by a number and letters so placed.

Thus, 7 . . . ab . . . $2b$, and $-4xyz$, are *simple quantities.*

16. A compound quantity is that which consists of two or more simple quantities, called its *terms*, connected together by the sign $+$ or $-$.

Thus $a + b$ is a *compound quantity*, having a and b for its *terms*; and $2x + 3y - 4$ is a *compound quantity*, the *terms* of which are $2x$, $3y$, and -4.

It is indifferent in what order the terms of a compound quantity stand, provided every term has its proper sign prefixed; thus, $a + b$ may be written $b + a$, and $2x + 3y - 4$ may be written $3y + 2x - 4$, or $-4 + 2x + 3y$, &c.

17. A binomial is a compound quantity consisting of two terms, as $a + z$ . . . $2b + 4c$, &c. A trinomial of three terms, as $x - 3y + 2z$, &c. A quadrinomial of four terms, as $-2a + b - 3c + x$. And a multinomial, or polynomial, of many terms, as $a - z + 26 - c + 4$, &c.

18. A residual quantity is a binomial having one of its terms negative, as $a - b$, or $2x - 5z$, &c.

19. The powers of a quantity are the products which arise by multiplying the quantity continually into itself; they are produced and named as follows.

Every quantity is considered as being the *first power* of itself; thus, a is the *first power* of a.

If a quantity be multiplied *once* into itself, the product is called the *second power*, or square of that quantity; thus $a \times a$, or aa, is the *second power* or *square* of a.

If a quantity be multiplied continually *twice* into itself, the product is called its *third power*, or *cube*; thus $a \times a \times a$, or aaa, is the *third power*, or *cube* of a.

In like manner any quantity, as a, multiplied three times into itself, produces $aaaa$, the *biquadrate*, or *fourth power* of a, and so on for higher powers.

20. But the powers of quantities are frequently and more conveniently represented by small figures, called *indices* or *exponents*, placed over, and a little to the right of, the quantities.

Thus a^2 is the same as aa, and denotes the *second power* or *square* of a . . . x^3 is the same as xxx, denoting the *third power* or *cube* of x . . . $yyyy$, or y^4, equally express the *fourth power* or

biquadrate of y, and so on, where the small figures 2, 3, and 4, are the *indices* or *exponents* of the powers, each shewing how often the quantity under it is repeated.

21. The root of a quantity is that which being multiplied once or oftener into itself, produces the given quantity.

Thus, a is the *square root* of a^2, because $a \times a = a^2$.
x is the *cube root* of x^3, because $x \times x \times x = x^3$.
y is the *fourth root* of y^4, because $y \times y \times y \times y = y^4$.

22. The root of a quantity is denoted by the character $\surd$, called a *radical sign*, with a small figure over it, expressing what root is designed[d]: or else, by a *fractional index* or *exponent* placed over the quantity.

Thus $\sqrt{a}$, or $a^{\frac{1}{2}}$, denotes the *square root* of a.
$\sqrt[3]{x}$, or $x^{\frac{1}{3}}$, denotes the *cube root* of x.
$\sqrt[4]{y}$, or $y^{\frac{1}{4}}$, denotes the *fourth root* of y, &c.

23. Quantities under the radical sign, or having a fractional index, are called *radical quantities*; if the root denoted by the radical sign, or index, can be found exactly, the quantity under it is called a *rational quantity*; but if the exact root cannot be found, it is then called an *irrational quantity*, or *surd*.

24. Like quantities are such as differ only in their numeral coefficients; thus $3a, \ldots 5a$, and $-17a$, are *like quantities*; so are $-4axy, \ldots 3axy$, and $-7axy$.

25. Unlike quantities are such as differ either in their letters, or indices, or in both; thus $2a$, and $3b$, are *unlike quantities*; so are $a^2x, \ldots ax^2$, and abx.

26. A vinculum is a straight line drawn over the top of two or more quantities, to connect them together, as $\overline{a + x}$, or $\overline{x - 3y + 4z}$, and signifies that the quantities under it are to be taken collectively, or considered as one quantity, with respect to the sign standing before or after the vinculum.

Thus $\overline{a + b} \times c$ signifies that both a and b are to be multiplied into c; ... $\overline{x + y + z} \times \overline{a - b}$ shews that the *sum* of x, y, and z, is to be multiplied into the difference of a and b.

---

[d] The square root being the *first* root, the small figure 2, denoting the root, is always omitted, and the root is designated simply by the sign $\surd$ placed before the power; but the figure denoting any higher root is never omitted.

27. The parenthesis is frequently used instead of the vinculum; thus $(a+b) \times c$ is the same as $\overline{a+b} \times c$, also $(x+y+z) \times (a-b)$ is the same as $\overline{x+y+z} \times \overline{a-b}$.

28. Two dots are sometimes used instead of the vinculum, or parenthesis; they are sometimes placed at each end of the compound quantity, as $: x+y-z : \times 4$; and sometimes at the end only, next the sign, as $x+y-z : \times 4$, which expressions are the same as $\overline{x+y-z} \times 4$.

29. When two quantities are included in a vinculum, or parenthesis, it is usual in reading them to pronounce the word *both*, and the word *all* when there are more than two: thus $\overline{a+b} \times c$ is read a plus b *both* into c; the expression $\overline{x-y} \div \overline{a+z}$, is read x minus y *both*, by a plus z *both*; $\overline{a-y+z}.\overline{x+a}$ is read a minus y plus z *all*, into x plus a *both*; and $(5+y-z) \div (a+b+c)$ is read 5 plus y minus z *all*, by a plus b plus c *all*.

30. When two quantities are compared together, the quantity compared is called the *antecedent*, and the quantity to which it is compared is called the *consequent*: also the relation the two quantities bear to each other with respect to magnitude, is called their *ratio*; and the quotient of the antecedent divided by the consequent, being the number expressing that relation, is called the *index of the ratio*: the ratio itself is expressed by placing the antecedent before the consequent, with two dots placed vertically between; thus $12 : 3$ expresses the *ratio* of 12 to 3, where 12 is the *antecedent*, 3 the *consequent*, and $\frac{12}{3} = 4$, the *index of the ratio*.

31. Proportion is the equality of two ratios: thus when two quantities have the same ratio that two other quantities have, this equality or identity of ratios is called proportion, and is denoted by four dots, placed in the interval between the first two and the latter two quantities; thus $3 : 6 :: 1 : 2$, is read 3 *is to* 6, *as* 1 *is to* 2, and denotes that 3 has the same ratio to 6, that 1 has to 2.

32. The sign $>$ is called *greater than*, and the sign $<$ *less than*; thus $a > b$ denotes that a is *greater than* b, and $b < a$ denotes that b is less than a.

33. The sign $\backsim$ denotes the difference in general of two quantities between which it is placed, when it is not known

which is the greater; thus $a \backsim b$ signifies $a - b$, when a is the greater; and $b - a$, when b is the greater.

34. The reciprocal of an integral quantity is unity divided by it, and the reciprocal of a fraction is that fraction inverted; thus the reciprocal of a is $\frac{1}{a}$, the reciprocal of $x - y$ is $\frac{1}{x-y}$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$, and that of $\frac{a-b}{c+d}$ is $\frac{c+d}{a-b}$.

35. EXAMPLES IN NOTATION,

Wherein the signification of each letter, and of the several combinations and results, are required to be expressed in numbers.

Let $a = 9$. $b = 7$. $c = 4$. $d = 3$. $e = 1$.

1. To find the value of $a + b + c + d$ in numbers?

Here, instead of a put 9, instead of b put 7, instead of c put 4, and instead of d put 3, with the proper signs between; then proceed with these numbers (viz. add them) as the signs import.

Thus $a + b + c + d = 9 + 7 + 4 + 3 = 23$, *the answer.*

2. Required the value of $a - b + c - d$?

Thus $a - b + c - d = 9 - 7 + 4 - 3 = 13 - 10 = 3$, *ans.*

Here I add the numbers with the sign + (viz. 9 and 4) together into one sum, and those with the sign — (viz. 7 and 3) into another, then subtract the latter from the former.

3. Required the value of $ab + bc - cd$?

Thus $ab + bc - cd = 9 \times 7 + 7 \times 4 - 4 \times 3 = 63 + 28 - 12 = 91 - 12 = 79$. *Answer.*

4. Required the value of $\frac{ab}{7d} + \frac{cd}{4e}$?

Thus $\frac{ab}{7d} + \frac{cd}{4e} = \frac{9 \times 7}{7 \times 3} + \frac{4 \times 3}{4 \times 1} = 3 + 3 = 6$. *Answer.*

5. What is the value of $\overline{a + b} \cdot \overline{c - e}$?

Thus $\overline{a + b} \cdot \overline{c - e} = \overline{9 + 7} \cdot \overline{4 - 1} = 16 \times 3 = 48$. *Ans.*

6. Required $\sqrt{a} - b^2 + c^3 - d + \frac{ad}{bc}$?

Thus $\sqrt{9} - \overline{7}|^2 + \overline{4}|^3 - 3 + \frac{9 \times 3}{7 \times 4} = 3 - 49 + 64 - 3 + \frac{27}{28} = 15\frac{27}{28}$. *Ans.*

7. What is the value of $a - c + b - 5$? *Ans.* 7.
8. To find the value of $2a + ab - d$? *Ans.* 78.
9. Required $3ab - 4cd - 12a$? *Ans.* 33.
10. To find $\overline{a + e} \times \overline{b - c} \times \overline{c - e}$? *Ans.* 90.
11. What is $abc \div bcd + cde$? *Ans.* 15.
12. Required $\frac{ab}{cd} + \frac{ac}{2e} - 2d$? *Ans.* $17\frac{1}{4}$.
13. Required $\sqrt{4a} - \sqrt{3d + c^2}$? *Ans.* 19.
14. Required the value of $\overline{d + e}|^2 \div c$? *Ans.* 4.
15. Required $a^2 - 2ab + b^2$? *Ans.* 4.
16. Find $\overline{a + b + 3d}|^{\frac{1}{2}} \times \overline{a - b + 6e}|^{\frac{1}{3}}$. *Ans.* 10.

ADDITION.

36. *To add quantities which are like, and have like signs.*

RULE. Add all the coefficients together, and place the common sign before their sum, and the common letter or letters after it[e].

EXAMPLES.

1. Add together the following algebraic quantities, viz.

$$\begin{array}{r} +\ 2a \\ +\ 3a \\ +\ 5a \\ +\ 4a \\ \hline +\ 14a \end{array}$$

Explanation.

I add the coefficients 2, 3, 5, and 4, together, and the sum is 14; before this I place the common sign +, and after it the common letter a, making the sum $+ 14a$. But, because when a quantity has no sign, + is always understood, the signs might in this case have been omitted without detriment.

---

[e] The reason of this rule is so exceedingly obvious, that he who can add numbers together will readily understand it; in the first example it is evident that $2a$, $3a$, $5a$, and $4a$, added together, will make $14a$, let a represent whatever it may; 2, 3, 5, and 4 times *any thing*, will (being added together) evidently amount to 14 times *that thing*: if a represent a pound, then the sum will be 14 pounds; if a represent a yard, then $14a$ will imply 14 yards. With respect to the signs, it is evident that the *sum* will be of the same nature with the *particulars* which constitute it; several *additive* quantities being collected together, the sum will clearly be an *additive* quantity equal to all of them together; but if several *subductive* quantities are collected, the sum will as evidently be a *subductive* quantity equal to all the latter taken together. Therefore the *sum* of affirmative quantities will be +, and the *sum* of negative ones —.

2.	3.	4.	5.	6.
$5ax$	$-4z$	$3a^2$	$-abc$	$2\sqrt{x}$
$2ax$	$-3z$	a^2	$-abc$	$3\sqrt{x}$
$3ax$	$-2z$	$4a^2$	$-2abc$	$5\sqrt{x}$
$2ax$	$-z$	$2a^2$	$-abc$	$\sqrt{x}$
$12ax$	$-10z$	$10a^2$	$-5abc$	$11\sqrt{x}$

7.	8.	9.	10.	11.
$-5az^3$	$4x$	$-2y$	$12ax$	$-\sqrt{abx}$
$-2az^3$	$2x$	$-2y$	$10ax$	$-2\sqrt{abx}$
$-az^3$	$4x$	$-3y$	$2ax$	$-3\sqrt{abx}$
$-az^3$	$3x$	$-4y$	$4ax$	$-2\sqrt{abx}$
$-9az^3$				

37. *To add quantities which are alike, and have unlike signs.*

RULE I. Add all the affirmative coefficients into one sum, and the negative ones into another.

II. Subtract the less of these sums from the greater, to the remainder prefix the sign of the greater, and subjoin the common letter or letters, as in the preceding rule [f].

---

[f] In order to explain this rule, it will be necessary to suppose a case. A trader in settling his books finds that several sums are due to him, and that he is indebted in several sums to others; to the former of these, (they being *additive*, or tending to *increase* his stock,) he prefixes for the sake of distinction the sign +, and to the latter, (these being *subductive*, or tending to *decrease* his stock,) he prefixes the sign −. Now in adjusting the account, he must add all the sums owing *to him* into one sum, and the sums *he owes* into another; he must then subtract the less of these sums from the greater, and the remainder will truly shew what he is worth. If the amount due *to* him be the greater, the remainder must have the sign + prefixed, and shews that he *has property* to that amount; if the sum of his debts be the greater, then the remainder will have the sign −, and shew that he is *so much in debt*. Now to apply this doctrine, (which is evident to the meanest capacity,) let us take the 12th example, where if x be considered as representing a pound sterling, $+7x$ will be 7 pounds, and $+8x$ eight pounds, both sums due *to* him; wherefore $-3x$ and $-2x$ will represent 3 pounds and 2 pounds, both sums due *by* him; now the sum of the two former, ($+7x$ and $+8x$,) or $15x$, viz. 15 pounds, will be his property, and the sum of the two latter, ($-3x$ and $-2x$,) or $-5x$, will be the sum of his debts, viz. 5 pounds; he therefore is worth 15 pounds, but has to pay 5 pounds out of this sum; the balance of his account then will be ($15x - 5x$, or) $10x$, or ten pounds, as in the example.

12. Add the following algebraic quantities together.

OPERATION.	*Explanation.*
$+\ 7x$ $-\ 3x$ $+\ 8x$ $-\ 2x$ $\overline{+\ 10x}$	I first add the affirmative quantities $+7x$, and $+8x$ together, making $+15x$; I then add the negative quantities $-3x$ and $-2x$ together, making $-5x$; next I subtract 5 from 15, and to the remainder 10 prefix the sign $+$ of the greater sum, and subjoin the common letter x, making $+10x$ for the sum: the sign $+$ might have been omitted.

13.	14.	15 [g].	16.	17.
$-2xz$	ab	z^2	$-\ abx$	$2a\sqrt{y}$
$+3xz$	$-ab$	$-\ 2z^2$	$8abx$	$a\sqrt{y}$
$-9xz$	$2ab$	$-\ 7z^2$	$-2abx$	$-\ 8a\sqrt{y}$
$+3xz$	$-ab$	$-12z^2$	$-6abx$	$9a\sqrt{y}$
$-5xz$	ab	$-20z^2$	$-\ abx$	$4a\sqrt{y}$

18.	19.	20.	21.	22.
$2z$	$-\ xy$	$3ax^4$	$-\ xyz$	$3\sqrt{z}$
$-3z$	$2xy$	$2ax^4$	$-2xyz$	$-\ \sqrt{z}$
$8z$	$5xy$	$7ax^4$	$-8xyz$	$-\ 4\sqrt{z}$
$-3z$	$-9xy$	$-4ax^4$	$5xyz$	$12\sqrt{z}$

38. *To add unlike quantities together* [h].

RULE I. Place like quantities under one another in a column,

That algebraic addition should sometimes require addition, and sometimes subtraction, appears at first sight an unaccountable paradox, but on an attentive examination the paradox vanishes; "it arises wholly from the scantiness of the name, from employing an old term in a new and enlarged sense:" if instead of *addition* you call this process *incorporation*, *striking a balance*, *union*, or any name sufficiently extensive to convey an adequate idea of the two-fold operation employed, the difficulty will be removed.

[g] In this example, if z^2 be supposed to represent a pound sterling, and if $+$ be prefixed to the sums *due to*, and $-$ to the sums *due by* any person, then if the example be considered as a true statement of his accounts, it appears that he is worth ($-20z^2$ or) -20 pounds; that is, he is 20 pounds in debt.

[h] Like quantities may be incorporated together, but unlike quantities cannot; we can add 2 pounds and 3 pounds together, and the sum is 5 pounds; but we cannot add 2 pounds and 3 shillings together, that is, we cannot incorporate them, so as to make either 5 pounds or 5 shillings: all that can be done is to write them down one after the other, thus, 2 pounds $+$ 3 shillings; or in the usual way, 2*l.* 3*s.* The same holds true of algebraic quantities; we cannot add b to a, so as to make either $2a$ or $2b$; all that can be done is to

each with its proper sign; and there will be as many columns as there are different kinds of quantities.

II. Add each column separately, and if the quantities in any column have *like signs*, add it by Art. 36. but if *unlike*, add it by Art. 37: the results, with their proper signs, placed in a line below, will be the sum required.

Note. It is usual to begin at the left hand column, and proceed from thence, in order, to the right.

23. Add the following algebraic quantities together; $2a + 3b - 4c + 5a + 2b - 7c + 3a - 2b + 8c + 2a + b - 7c$.

OPERATION.

$$\begin{array}{r} 2a + 3b - 4c \\ 5a + 2b - 7c \\ 3a - 2b + 8c \\ 2a + b - 7c \\ \hline 12a + 4b - 10c \end{array}$$

Explanation.

I place all the a's in the first column, and because they are all $+$, and likewise the leading quantities, I omit the sign; I then place all the b's with their proper signs in the second column, and all the c's in the third; I next add up the first, or left hand column, by Art. 36, and place the sum $12a$ below: I then add the second and third columns by Art. 37, the sum of the former being $+4b$, and that of the latter $-10c$; these three sums placed in a line, with their proper signs, are the answer.

24. Add $3x + 5y - 6z - 2x - 8y - 9z + 20x + 2y - 3z + x - y + z - 4$ together.

OPERATION.

$$\begin{array}{r} 3x + 5y - 6z \\ -2x - 8y - 9z \\ 20x + 2y - 3z \\ x - y + z - 4 \\ \hline 22x - 2y - 17z - 4 \end{array}$$

Explanation.

Here I place the x's in one column, the y's in another, the z's in another, and the -4 in another; then I add by Art. 37, and place the results, with their proper signs, in a line below, for the sum.

Add the following algebraic quantities together, viz.

25.	26.	27.
$2a - 8x$	$-4y + 5$	$3 - 2y + z$
$-3a + x$	$12y - 8$	$5 + 4y - 2z$
$-4a - 2x$	$-3y + 2$	$2 - y - z$
$-a + 4x$	$2y + 4$	$-10 - y + 2z$
$-6a - 5x$	$7y + 3$	0

---

place them one after the other, each with its proper sign, thus $a + b$. And when there are several quantities concerned, to collect all that are of one kind into one sum, and all those of another kind into another, &c. by the former rules, and then write down the several sums in succession, each with its proper sign.

28.

$$\begin{array}{r} ax - xy - z \\ 2ax + xy + 2z \\ -4ax - xy + 3z \\ ax - xy \quad * \quad -2 \\ \hline \end{array}$$

29.

$$\begin{array}{r} 4b - 5c \\ 3b + 6c \\ 2b - 7c \\ 4b + 5c \\ \hline \end{array}$$

30.

$$\begin{array}{r} x\sqrt{z} + 4 \\ -x\sqrt{z} - 8 \\ 2x\sqrt{z} + 5 \\ -x\sqrt{z} + 2 \\ \hline \end{array}$$

31.

$$\begin{array}{r} 2 + 4ax \\ 3 - 5ax \\ -2 + ax \\ -1 - 9ax \\ \hline \end{array}$$

32.

$$\begin{array}{r} a^2 - 2ax \\ a^2 + 3ax + 5 \\ -\ a^2 - 4ax - 10 \\ 3a^2 + 5ax \\ \hline \end{array}$$

33. Add $4a + 5b + 6a - 7b + 8a - 9b + 2a - 2b$ together. *Sum* $20a - 13b$.

34. Add $3x - 5y + 2z - 2x + 4y - 8z + 2x + 3y + z - x + y - z$ together. *Sum* $2x + 3y - 6z$.

35. Add $2 - x + 4y + 3 + 3x - y - 30 - x - 2y + 1 - 2x + 3y - 10z$ together. *Sum* $24 - x + 4y - 10z$.

36. Add $2\sqrt{a} - bc + 3\sqrt{a} + 4bc - \sqrt{a} - 5bc + 2\sqrt{a} - bc + d - 8$ together. *Sum* $6\sqrt{a} - 3bc + d - 8$.

39. *To add quantities under a vinculum*[1].

RULE. I. If the quantities *under the vinculum* are alike, viz. if they consist of the same letters, numbers, coefficients, signs, and indices, they are added by taking the sum of the coefficients *without the vinculum,* and subjoining the vinculum, with the quantities under it, to this sum.

II. But if the quantities under the vinculum are unlike, (viz.

---

[1] The vinculum connects all the quantities included under it into one, (Art. 26.) they are consequently to be managed collectively like a simple quantity. When the quantities under the vinculum are in all respects alike, (namely, in *signs, letters,* and *indices,*) the addition will be performed by adding the coefficients, (or numbers connected with, but not under, the vinculum,) and subjoining the common vinculum to the sum, as is shewn in Art. 36. 37.

But when the quantities under the vinculum are not *in all respects* alike, (namely, if they differ either in the *signs, letters,* or *indices,*) they are evidently unlike quantities, and admit of no other addition than merely connecting them by the signs of their coefficients, as in Art. 38.

if they differ in any of those particulars,) they can only be connected together by their proper signs.

37. Add $3\sqrt{x+y} + 4\sqrt{x+y} + 5\sqrt{x+y} + \sqrt{x+y}$ together.

OPERATION.

$$\begin{array}{r} 3\sqrt{x+y} \\ 4\sqrt{x+y} \\ 5\sqrt{x+y} \\ \sqrt{x+y} \\ \hline \textit{Sum}\ 13\sqrt{x+y} \end{array}$$

Explanation.

The quantities under the vinculum being alike in all respects, I merely add the coefficients 3, 4, 5, and 1, (understood,) together, by Art. 36. and to their sum 13 subjoin the vinculum for the answer.

38. Add $-3.\overline{y-4}|^{\frac{1}{3}} + 2.\overline{y-4}|^{\frac{1}{3}} - 9.\overline{y-4}|^{\frac{1}{3}} + 8.\overline{y-4}|^{\frac{1}{3}}$ together.

OPERATION.

$$\begin{array}{r} -3.\overline{y-4}|^{\frac{1}{3}} \\ 2.\overline{y-4}|^{\frac{1}{3}} \\ -9.\overline{y-4}|^{\frac{1}{3}} \\ 8.\overline{y-4}|^{\frac{1}{3}} \\ \hline \textit{Sum}\ -2\ \overline{y-4}|^{\frac{1}{3}} \end{array}$$

Explanation.

The quantities under the vinculum being alike, I add the coefficients -3 and -9 into one sum, and 2 and 8 into another; subtract the less from the greater, and to the remainder 2 prefix the sign $-$ (Art. 37.) and subjoin the vinculum.

39. Add $2\sqrt{y-z} + 3\sqrt{x+z} - \sqrt[3]{2y-x^2} - 2\sqrt{y-z}$ together.

OPERATION.

$$\begin{array}{r} 2\sqrt{y-z} \\ 3\sqrt{x+z} \\ -\sqrt[3]{2y-x^2} \\ -2\sqrt{y-z} \\ \hline \textit{Sum}\quad 3\sqrt{x+z} - \sqrt[3]{2y-x^2} \end{array}$$

Explanation.

The first and last of these quantities destroy each other; the remaining two being unlike, can be added only by connecting them by their proper signs.

40. Add $2a + 3\sqrt{x+1} + 2a + 4\sqrt{x+1} - a + \sqrt{x+1} + 3a - 2\sqrt{x+1}$ together. *Sum* $6a + 6\sqrt{x+1}$.

41. Add $ax + 2b^2 - b + \overline{a+b}|^{\frac{1}{2}} - 3ax + b^2 - 4b + 12ax - 8b^2 - 3b + 2.\overline{a+b}|^{\frac{1}{2}}$ together. *Sum* $10ax - 5b^2 - 8b + 3.\overline{a+b}|^{\frac{1}{2}}$.

42. Add $a\sqrt{b} + \sqrt{x+y+z} + 4\sqrt{x+y} - 3a\sqrt{b} + xyz - \sqrt{x+y+z} - 2\sqrt{a} + 2a\sqrt{b}$ together. *Sum* $4\sqrt{x+y} + xyz - 2\sqrt{a}$.

43. Add $x + y - z - 2ax\sqrt{3} - x + 12 + z + ax\sqrt{3} - 10 - y + ax\sqrt{3} - 2$ together.

40. SUBTRACTION[k].

RULE. I. Place those quantities from which the subtraction is to be made, in one line, and the quantities to be subtracted, in a line below them; so that like quantities may stand exactly under one another.

II. Change the signs of all the quantities to be subtracted, or conceive them to be changed; then add them, (so changed,) each to its like, in the upper line, by the rules in Addition; the result will be the difference required.

III. When any quantities occur in the upper line, which have not their like in the lower, those quantities must be put down to the difference, *with their proper signs:* and when any occur in the lower line, which have not their like in the upper, they must be put down *with their signs changed.*

---

[k] Subtraction being the converse of addition, its method of operation will evidently be the converse of that employed in addition: it proceeds on this principle, namely, to subtract an affirmative quantity, is the same as to add a negative one of equal value; and to subtract a negative quantity, is the same as to add an equivalent affirmative quantity: thus, if the pounds which increase my property be called *affirmative*, then will those which decrease it, (namely, the pounds which constitute my debts,) be *negative*. Now to take away (or pay) a debt of 1 pound, is the same as to add 1 pound to my property; and to take away 1 pound from my property, is just the same as to add a debt of 1 pound to my account: if this be well understood, the reason of the rule will be obvious, from what has been said in the note on the second case of addition. That this rule sometimes requires addition to be used, and sometimes subtraction, shews that the name *subtraction* is not sufficiently comprehensive to express the nature of the two-fold operation implied by the rule; the business of which is to find a quantity, which being algebraically incorporated with either of two given quantities, the result will be equal to the other given quantity.

Mr. Thomas Simpson accounts for the rule in this manner: let $-3a+2b$ be subtracted from $4a-6b$, as in the first example. Now it is plain that if the affirmative part, namely $2b$, were alone to be subtracted, the remainder would in that case be ($4a-6b-2b$, or) $4a-8b$; but since the quantity to be subtracted, namely, $-3a+2b$, is less than $2b$ by the quantity $3a$, too much has been taken away by the said $3a$; we must therefore add $3a$ to the remainder, to make it what it ought to be; that is, instead of $4a-8b$, the remainder, thus corrected, will be ($4a-8b+3a$, or) $7a-8b$, as in the example.

EXAMPLES.

1. From $4a - 6b$ subtract $-3a + 2b$.

OPERATION.

$$\begin{array}{r} 4a - 6b \\ \underline{-3a + 2b} \\ \underline{7a - 8b} \end{array}$$

Explanation.

I place the latter quantities under the former: then the quantities to be subtracted being $-3a + 2b$, I change their signs, and they become $+3a - 2b$; these I add, thus, $+3a$ to $+4a$, and the sum is $+7a$ to put down; $-2b$ to $-6b$, and the sum is $-8b$ to put down, by Art. 36.

2. From $3x - 5y$ subtract $2x - 7y$.

OPERATION.

$$\begin{array}{r} 3x - 5y \\ \underline{2x - 7y} \\ \underline{x + 2y} \end{array}$$

Explanation.

Changing the signs of the lower quantities, they become $-2x + 7y$; then $-2x$ added to $+3x$, gives $+x$, to put down, and $+7y$ added to $-5y$ gives $+2y$, to put down, by Art. 37.

3. From $a - ax$ take $-5a - z$.

OPERATION.

$$\begin{array}{r} a - ax \phantom{{}-z} \\ \underline{-5a \quad * \quad - z} \\ \underline{6a - ax + z} \end{array}$$

Explanation.

Here $-ax$ is not in the lower line, and $-z$ is not in the upper; therefore I put the former down, *with its proper sign*, and the latter *with its sign changed*, according to the rule.

Required the difference of the following quantities.

4.

$$\begin{array}{r} 5x - 2y \\ \underline{-4x + 3y} \\ \underline{9x - 5y} \end{array}$$

5.

$$\begin{array}{r} 4b - 2d \\ \underline{2b - d} \\ \underline{2b - d} \end{array}$$

6.

$$\begin{array}{r} 3x + \sqrt{z} \phantom{{}-2} \\ \underline{3x + 4\sqrt{z} - 2} \\ \underline{* - 3\sqrt{z} + 2} \end{array}$$

7.

$$\begin{array}{r} ax - 2x^2 + 3 \\ \underline{-\ ax + 2x^2 \phantom{{}+3}} \\ \underline{} \end{array}$$

8.

$$\begin{array}{r} 2y + 8 \\ \underline{y + 4} \\ \underline{} \end{array}$$

9.

$$\begin{array}{r} -1 + 8z \\ \underline{2 - 2z} \\ \underline{} \end{array}$$

10.

$$\begin{array}{l} ay - y^2 + 7 \\ \underline{ay - y^2 \phantom{{}+7}} \\ \underline{} \end{array}$$

11.

$$\begin{array}{l} 3d + 5z^3 \\ \underline{4d - 6z^3 + 5z} \\ \underline{} \end{array}$$

12. From $3x - 4z$ take $2x - 6z$. *Diff.* $x + 2z$.

13. From $ax - x$ take $-2ax - y$. *Diff.* $3ax - x + y$.

14. From $3z - 1$ take $-4a + b$. *Diff.* $3z - 1 + 4a - b$.

41. *To subtract quantities under the vinculum.*

RULE I. If the quantities under the vinculum are in all respects alike, change the sign of the coefficient of the lower vinculum, then add it to that of the upper, and to the result subjoin the common vinculum, &c.

II. But if they are not alike, put all the quantities down in succession, with the sign of the lower coefficient changed[1].

15. From $14\sqrt{a+x}$ take $-9\sqrt{a+x}$.

OPERATION.

$$\begin{array}{r} 14\sqrt{a+x} \\ -\ 9\sqrt{a+x} \\ \hline 23\sqrt{a+x} \end{array}$$

Explanation.

The quantities under the vinculum being alike, I only change the sign of the lower coefficient, (-9 to $+9$,) then add it to the upper, and to the sum subjoin the common vinculum.

16. From $12.\overline{ax-z}|^{\frac{2}{3}}$ take $15.\overline{ax-z}|^{\frac{2}{3}}$.

OPERATION.

$$\begin{array}{r} 12.\overline{ax-z}|^{\frac{2}{3}} \\ 15.\overline{ax-z}|^{\frac{2}{3}} \\ \hline -3.\overline{ax-z}|^{\frac{2}{3}} \end{array}$$

Explanation.

The quantities under the vinculum are alike; I change the sign of the 15 from $+$ to $-$, then adding -15 to $+12$, the sum is -3, to which I subjoin the vinculum, &c.

17. From $-3\sqrt{a-z}$ take $2\sqrt{a+x}$.

OPERATION.

$$\begin{array}{l} -3\sqrt{a-z} \\ \ \ \ 2\sqrt{a+x} \\ \hline -3\sqrt{a-z}-2\sqrt{a+x} \end{array}$$

Explanation.

The quantities under the vinculum being unlike, I only change the sign of the lower coefficient from $+$ to $-$, and put all the quantities down in succession.

18. From $5\sqrt{x-y}$ take $2\sqrt{x-y}$. *Diff.* $3\sqrt{x-y}$.

19. From $2x-\sqrt{1+x}$ take $-x+\sqrt{1+x}$. *Diff.* $3x-2\sqrt{1+x}$.

20. From $4\sqrt{a+y}$ take $-2\sqrt{a+y}-ax$. *Diff.* $6\sqrt{a+y}+ax$.

21. From $az-2\sqrt{a-2}$ take $-3az+x-\sqrt{a-2}$. *Diff.* $4az-x-\sqrt{a-2}$.

---

[1] The observations contained in the note on Art. 39. are applicable to this rule, and therefore need not be repeated.

22. From $a^2 - 2\,ax + x\,\sqrt{x - 2\,y}$ take $3\,x\,\sqrt{x - 2\,y} + 1$. *Diff.* $a^2 - 2\,ax - 2\,\sqrt{x - 2\,y} - 1$.

23. From $3\,b - 4 + \sqrt{a + b}$ take $12 - x + 3.\overline{a + b}|^2$. *Difference* $3\,b - 16 + x + \sqrt{a + b} - 3.\overline{a + b}|^2$.

24. From $x^2 - 3\,xyz - y^3$ take $x^2 + 3\,xyz - y^2 + \overline{4 - a}|^3$. *Diff.* $- 6\,xyz - y^3 + y^2 - \overline{4 - a}|^3$.

MULTIPLICATION.

To multiply simple quantities together.

42. *When the factors have the same letters in each;*

RULE I. If the factors have *like signs,* (viz. both +, or both −,) make the sign of the product +; but if they have unlike signs, (viz. one + and the other −,) make the sign of the product −.

II. Multiply both coefficients together, and the result will be the coefficient of the product.

III. Add the index of each letter in one factor, to the index of the same letter in the other, and the sum will be the index, which must be placed over that letter in the product.

IV. Place the sign, coefficient, and letters, as found above, in order, and it will be the product required [m].

---

[m] The rule for the numeral coefficients is plain from the nature of arithmetical multiplication; that for the indices follows from the method of notation: thus, suppose a^3 is to be multiplied into a^2, the product will be a^5, or a with the index 5, which rises from adding the indices 3 and 2 together: for $a^3 = aaa$, and $a^2 = aa$, (Art. 20.) therefore $a^3 \times a^2 = aaa \times aa$, or $aaaaa$ (by Art. 20.) but this expression equals a^5 (Art. 20.) therefore $a^3 \times a^2 = a^5$; or the multiplication of like algebraic quantities is performed by adding the indices of like letters in both factors together, and placing the sum as an index over its letter in the product.

That like signs in the factors give *plus* in the product, and unlike signs in the factors give *minus* in the product, is thus explained.

1. When $+ a$ is to be multiplied into $+ b$, it is implied, that $+ a$ is to be taken as many times as there are units in b; that is, as many a's are to be added together, as b contains units; and since the sum of any number of affirmative terms is affirmative, it follows, that $+ a$ taken $+ b$ times is affirmative, or that $+ a \times + b = + ab$.

2. When quantities are to be multiplied together, it is indifferent in what

EXAMPLES.

1. Multiply $4a^3$ into $5a^2$.

$$\begin{array}{l} 4\,a^3 \\ \underline{5\,a^2} \\ \underline{20\,a^5} \text{ product} \end{array}$$

Explanation.

The signs of both factors being +, (understood,) that of the product will be + (understood;) then $4 \times 5 = 20$ for the coefficient, and $2+3=5$ for the index; wherefore $20\,a^5$ is the product required.

Or thus,

$$4\,a^3 \times 5\,a^2 = 20\,a^5 \text{ product.}$$

---

order they are placed, for $a \times b$ is the same as $b \times a$; therefore when $-a$ is to be multiplied into $+b$, or $+b$ into $-a$, this is the same as taking $-a$ as many times as there are units in $+b$; and since the sum of any number of negative terms is evidently negative, it follows that $-a$ times $+b$, or $+a$ times $-b$ is negative, that is, $-a \times +b$, or $+a \times -b$, will each produce $-ab$.

3. When $-a$ is to be multiplied into $-b$, it is implied that $-a$ is to be *subtracted* (not added) as often as there are units in b, because the sign $-$ denotes subtraction; but subtracting a negative quantity, is the same as adding an affirmative one of equal value, (see the note on Art. 40.) consequently $-a$ subtracted b times, is the same as $+a$ added b times; that is, $-a \times -b$ is the same as $+a \times +b$, which produces $+ab$, as has been shewn in the former part of this note. Therefore + multiplied by +, and $-$ multiplied by $-$, each produces +; also + multiplied by $-$, or $-$ multiplied by +, produces $-$.

The same may be shewn otherwise; thus, it is evident, that if a compound quantity equal to *nothing*, be multiplied by any quantity whatever, the *product* will be *nothing*. Now since $a-a=0$, let this be multiplied by $+b$, the first term of the product will evidently be $+ab$, (by the former part of the note,) wherefore the second term of the product must be $-ab$, otherwise the product of $a-a=0$ multiplied by $+b$, (or 0 multiplied by b,) will not be equal to nothing, which is absurd; wherefore if $+a \times +b = +ab$, then will $-a \times +b = -ab$.

Let now $a-a=0$ be multiplied by $-b$; the first term of the product being $-ab$, from what has just now been shewn, the second term must of course be $+ab$, to make the product $=0$: therefore $-a \times -b = +ab$.

That these conclusions are true, appears from their application; let $8-4=4$, be multiplied into $5-3=2$.

$$\begin{array}{l} 8-4 \\ \underline{5-3} \\ 40-20 \\ \underline{\quad -24+12} \\ \underline{40-44+12} = 52-44 = 8, \text{ but } 4 \times 2 = 8 \text{ also;} \end{array}$$

Wherefore $\overline{8-4} \times \overline{5-3} = 4 \times 2$, or the product of the two compound quantities, (observing the above rule for the signs,) equals the product of their equivalent simple numbers found by common multiplication: which was to be shewn.

2. Multiply $4\,a^2x^3y$ into $-\,2\,a^2x^3y^2$.

OPERATION.	*Explanation.*
$4\,a^2\,x^3\,y$ $-\,2\,a^2\,x^3\,y^2$ $-\,8\,a^4\,x^6\,y^3$ *prod.*	The signs being unlike, that of the product is $-$, then $2 \times 4 = 8$, the coefficient; and $2+2=4$, the index of a; $3+3=6$, the index of x; and $2+1$ (understood) $=3$, the index of y; wherefore $-\,8\,a^4\,x^6\,y^3$ is the product.

Or thus,

$4\,a^2x^3y \times -2\,a^2x^3y^2 = -8\,a^4x^6y^3$. *Product.*

3. Multiply $-\,2\,ab^2c$ into $3a^3bc$.

Thus, $-\,2\,ab^2c \times 3\,a^3bc = -\,6\,a^4b^5c^2$, *the product.*

4. Multiply $-\,2\,ax^2$ into $-\,5\,ax^3$.

Thus $-\,2\,ax^2 \times -\,5\,ax^3 = 10\,a^2x^5$, *product.*

5. Multiply $3\,x^2y$ into $8\,x^2y^2$. *Prod.* $24\,x^4y^3$.

6. Multiply xyz^2 into $-\,x^2yz$. *Prod.* $-\,x^3y^2z^3$.

7. Multiply $-\,4\,ab^2$ into $-\,3\,a^2b^2$. *Prod.* $12\,a^3b^4$.

43. *When the factors consist of different letters.*

RULE. Find the sign and coefficient of the product as before, and to the coefficient subjoin all the letters in both factors for the product[a].

8. Multiply $3ab$ into $-\,5\,xy$.

OPERATION.	*Explanation.*
$3\,ab$ $-\,5\,xy$ $-\,15\,abxy$, *prod.*	The signs being unlike, that of the product will be $-$, then $3 \times 5 = 15$ the coefficient; then putting down the sign, coefficient, and all the letters, (in alphabetical order,) we have the product required.

9. Multiply $-\,3\,xy$ into $-\,7\,az$.

Thus $-\,3\,xy \times -\,7\,az = 21\,axyz$, *the product.*

10. Multiply $4\,ab^2$ into $3\,b^2xyz^3$.

Thus $4\,ab^2 \times 3\,b^2xyz^3 = 12\,ab^4xyz^3$. *Product.*

11. Multiply abc into xyz. *Prod.* $abcxyz$.

12. Multiply $-\,ax$ into $20\,xy$. *Prod.* $-\,20\,ax^2y$.

13. Multiply $-\,2\,bz$ into $-\,4\,a^3xz$. *Prod.* $8\,a^3bxz^2$.

44. *When one of the factors is a compound quantity.*

RULE I. Place the simple quantity under the left hand place of the compound one; or place both factors in a line, with the

[a] This rule is evident from the nature of algebraic Notation, Art. 10. It is most convenient to place the letters alphabetically.

sign × between them, and a vinculum over the compound quantity.

II. Multiply each of the terms of the compound quantity by the simple one, by the foregoing rules; and place the several products, with their proper signs, in a line; these will form the product required*.

14. Multiply $2a - 3x$ by $4b$.

OPERATION.

$$\begin{array}{l} 2a \;-\; 3x \\ 4b \\ \hline 8ab - 12bx \text{ product.} \end{array}$$

Explanation.

$2a$ multiplied by $4b$, gives $8ab$, by Art. 43. and $-3x$ multiplied by $4b$, gives $-12bx$; therefore $8ab - 12bx$ is the product required.

Or thus,

$\overline{2a - 3x} \times 4b = 8ab - 12bx$, *product.*

15. Multiply $3ax - 2b^2 + z$ into ab.

OPERATION.

$$\begin{array}{l} 3ax \;\; -2b^2 \;\; +z \\ \;\; ab \\ \hline 3a^2bx - 2ab^3 + abz \text{ product.} \end{array}$$

Explanation.

$3ax \times ab = 3a^2.bx \ldots -2b^2 \times ab = -2ab^3$ and $z \times ab = abz$; these products placed in order, with their proper signs, give the product required. Art. 43.

Or thus,

$\overline{3ax - 2b^2 + z} \times ab = 3a^2bx - 2ab^3 + abx$, *product.*

Multiply the following quantities together.

16.

$$\begin{array}{l} 3x \;-2y \\ 4x \\ \hline 12x^2 - 8xy \end{array}$$

17.

$$\begin{array}{l} 2a \;\; +3b \;\; -2 \\ -2ax \\ \hline -4a^2x - 6abx + 4ax \end{array}$$

18.

$$\begin{array}{l} 4a \;\; -x^2y \;\; +xy^2 \;\; -3 \\ -\;ay \\ \hline -4a^2y + ax^2y^2 - axy^3 + 3ay \end{array}$$

19.

$$\begin{array}{l} 2x - 3y \\ 4z \\ \hline \end{array}$$

---

* This rule follows from the preceding; for the product of the whole multiplicand into the multiplier, is evidently equal to the sum of the products of the several parts of the former, multiplied into the latter; the truth of which is likewise demonstrated geometrically, in the first proposition of the second book of Euclid's Elements.

20.	21.	22.
$3a^3+2b$	$5-8z$	$a^2+2ax-b$
c	$-3ay$	$-2a^2$

23. Multiply $2ay-ay^2+3ay^3$ into $4by$. *Prod.* $8aby^2-4aby^3+12aby^4$.

24. Multiply $-a^2+2x-6$ by xy. *Prod.* $-a^2xy+2x^2y-6xy$.

25. Multiply $x^2-2xz+z^2$ into -4. *Prod.* $-4x^2+8xz-4z^2$.

26. Multiply $3-abx+2ab$ into 12. *Prod.* $36-12abx+24ab$.

45. *When both factors are compound quantities.*

RULE I. Place the multiplier under the multiplicand, and let them both stand in the same order.

II. Multiply the first or left hand term of the multiplier, into every term of the multiplicand, and place the products, with their proper signs, in a line below, by the preceding rule.

III. Multiply the second term of the multiplier into every term of the multiplicand; and place the products, with their proper signs, under the former products, observing to set like quantities (when they occur) under each other.

IV. Proceed in this manner with every term of the multiplier, then add all the products together, and the sum will be the product required [p].

27. Multiply $2a+3b$ into $4a+5b$.

OPERATION.

$$
\begin{array}{rrrr}
 & 2a & +\ 3b & \\
 & 4a & +\ 5b & \\
\hline
 & 8a^2 & +\ 12ab & \\
 & & +\ 10ab & +\ 15b^2 \\
\hline
\textit{Prod.} & 8a^2 & +\ 22ab & +\ 15b^2 \\
\hline
\end{array}
$$

Explanation.

I first multiply the upper line by $4a$, and the product $8a^2+12ab$ is the first line of the product. I then multiply the top line by $5b$, and the product $10ab+15b^2$ is the second line of the product; likewise I place the ab's under each other. I then add these two products together, and the sum is the last line, or product required.

---

[p] Here the whole multiplicand is considered as multiplied into the whole multiplier, when every member of the one is multiplied into every member of the other, and their products collected together, (as far as is practicable,) into one sum: the truth of this follows from what has been said in the note on the preceding rule, and it may be illustrated and confirmed by particular examples in numbers, such as the example given at the end of the note on Art. 42.

28. Multiply $x+2y$ into $3x-4y$.

OPERATION. *Explanation.*

$$\begin{array}{lll}
x+2y & & \\
3x-4y & & \\
3x^2+6xy \ldots\ldots\ldots & =\overline{x+2y}\times & 3x \\
\quad -4xy-8y^2 \ldots\ldots & =\overline{x+2y}\times & -4y \\
\textit{Prod.}\ 3x^2+2xy-8y^2 \ldots\ldots & =\overline{x+2y}\times & \overline{3x-4y}
\end{array}$$

Multiply the following quantities together.

29.

$$\begin{array}{l}
a+z \\
a+z \\
\hline
a^2+\ az \\
\qquad az+z^2 \\
\hline
a^2+2az+z^2
\end{array}$$

30.

$$\begin{array}{l}
x+y \\
x-y \\
\hline
x^2+xy \\
\quad -xy-y^2 \\
\hline
x^2\ \ *\ -y^2
\end{array}$$

31.

$$\begin{array}{l}
a^2-2ab+b^2 \\
a-b \\
\hline
a^3-2a^2b+\ ab^2 \\
\quad -\ a^2b+2ab^2-b^3 \\
\hline
a^3-3a^2b+3ab^2-b^3
\end{array}$$

32.

$$\begin{array}{l}
5x-4z \\
2x+z \\
\hline
10x^2-8xz \\
\qquad +5xz-4z^2 \\
\hline
10x^2-3xz-4z^2
\end{array}$$

33.

$$\begin{array}{l}
x-8 \\
x-4 \\
\hline
x^2-\ 8x \\
\quad -\ 4x+32 \\
\hline
x^2-12x+32
\end{array}$$

34.

$$\begin{array}{l}
y^2+yz+z^2 \\
y-z \\
\hline
y^3+y^2z+yz^2 \\
\quad -y^2z-yz^2-z^3 \\
\hline
y^3\ \ *\ \ *\ -z^3
\end{array}$$

35.

$$\begin{array}{l}
a+b+c \\
x-y-z \\
\hline
ax+bx+cx-ay-by-cy-az-bz-cz
\end{array}$$

36. Multiply $a+x$ into $a+x$. *Prod.* $a^2+2ax+x^2$.

37. Multiply $2x+3y$ into $3x-4y$. *Prod.* $6x^2+xy-12y^2$.

38. Multiply $x+1$ into $x-1$. *Prod.* x^2-1.

39. Multiply $3+z$ into $4-z$. *Prod.* $12+z-z^2$.

40. Multiply $2a^2-3a-4$ into $3a^3-5a$. *Prod.* $6a^5-9a^4-22a^3+15a^2+20a$.

41. Multiply $2x^2+3xy-5y^2$ into x^2-y^2. *Prod.* $2x^4+3x^3y-7x^2y^2-3xy^3+5y^4$.

42. Multiply $4x-5a-2b$ into $3x-2a+5b$. *Prod.* $12x^2-23ax+14bx+10a^2-21ab-10b^2$.

DIVISION.

To divide one simple quantity by another.

46. *When both terms consist of the same letters, and the dividend contains the divisor some number of times exactly.*

RULE I. Place the dividend above the divisor, with their proper signs, and a small line between them, like a fraction.

II. If the terms have *like signs*, make the sign of the quotient +; but if they have *unlike signs*, make that of the quotient —.

III. Divide the coefficient of the upper term by that of the lower, and the result will be the coefficient of the quotient [q].

---

[q] The rule for the coefficients is the same as that for simple division in Arithmetic. (Art. 37. Part I.)

With respect to the subtraction of the index of the divisor from that of the dividend, in order to find the index of the quotient, it may be observed, that division being the converse of multiplication, the methods of operation in one will evidently be the converse of those in the other. Wherefore, since it has been shewn, (Art. 42. and *note*,) that addition of the indices of like letters produces multiplication, it follows, that subtracting the indices (viz. that of the divisor from that of the dividend) will produce division; or in other words, if the index of the divisor be subtracted from that of the dividend, the remainder will be the index of the quotient.

If one quantity be divided by another without remainder, it is plain that the quotient will be such, that being multiplied by the divisor, the resulting product will equal the dividend; now it is equally plain that the sign of the quotient must be such, that when the quotient is multiplied by the divisor, the product will (not only be equal to, but) have the same sign with the dividend, according to the rule for the signs in Art. 42. Wherefore it follows, that

1. When both terms (namely the divisor and dividend) are +, the quotient must likewise be +; for + in the divisor must have + in the quotient, to produce + in the dividend.

2. When both terms are —, the quotient must be +; for — in the divisor must have + in the quotient, to produce — in the dividend.

3. When either of the terms is +, and the other —, the quotient must be —; for + in the divisor must have — in the quotient, to produce — in the dividend.

And — in the divisor must have — in the quotient, to produce + in the dividend.

IV. Subtract the index of each letter in the lower term, from the index of the same letter in the upper, and place the remainder as an index over that letter in the quotient.

V. Having proceeded in this manner with all the letters concerned, the result, connected with the sign and coefficient (found as above) will be the quotient required.

If any letter has the same index in both terms, that letter is cancelled from the operation, and does not come into the quotient.

Examples.

1. Divide $12\,a^6x^8$ by $4a^3x^4$.

Operation.

$$\frac{12\,a^6x^8}{4\,a^3x^4}=3\,a^3x^4 \textit{ quot.}$$

Explanation.

Having placed the terms, I find that they have *like signs*; wherefore that of the quotient is + understood: next I divide 12 by 4, and the quotient 3 is the coefficient of the quotient: then I subtract 3, the index of a in the lower term, from 6, the index of a in the upper, and the remainder 3 is the index of a in the quotient. I do the same by x, subtracting 4 from 8, and the remainder 4 is in like manner the index of x in the quotient.

2. Divide $56x^4y^6$ by $-7x^2y^2$.

Operation.

$$\frac{56\,x^4y^6}{-7\,x^2y^2}=-8\,x^2y^4 \textit{ quot.}$$

Explanation.

Unlike signs here give $-$, and $56 \div 7 = 8$ the coefficient; also $4-2=2$ the index of x, and $6-2=4$ the index of y.

3. Divide $28\,a^6z^4$ by $-7\,ax^2$, and $-8x^3$ by $2x$.

$$\frac{28\,a^6z^4}{-7\,ax^2}=-4\,a^5z^2. \textit{ quot.} \qquad \frac{-8\,x^3}{2\,x}=-4\,x^2. \textit{ quot.}$$

4. Divide $2\,y^5z^4$ by $2\,y^3z$, and $-xy^2z^3$ by $-xyz$.

$$\frac{2\,y^5z^4}{2\,y^3\,z}=y^2z^3. \qquad \frac{-xy^2z^3}{-xyz}=yz^2.$$

The following is a summary of the whole doctrine brought into one point of view.

	Divisor.	*Dividend.*	*Quotient.*
1.	+)	+	(+
2.	−)	−	(+
3.	{ −)	+	(−
	{ +)	−	(−

Wherefore in division (as in multiplication) *like signs always produce* +, *and unlike* −.

5. Divide $4x^5$ by $2x^3$. *Quot.* $2x^2$.

6. Divide $2a^2b^2c^3$ by $-abc$. *Quot.* $-2abc^2$.

7. Divide $-100abcx^2y^3$ by $-5abcxy$. *Quot.* $20xy^2$.

47. *When the dividend contains more letters than the divisor, and is exactly divisible by it.*

RULE. Proceed with the coefficients, and the letters that are alike in both terms, by the last rule, and place the remaining letters of the dividend in the quotient [r].

8. Divide $-8a^3bc$ by $-2a$.

OPERATION.

$$\frac{-8a^3bc}{-2a}=4a^2bc.$$

Explanation.

First, like signs give + understood, and $3-1=2$ the index of a; secondly, I subjoin the superfluous letters bc to the part $4a^2$, making $4a^2bc$ for the quotient.

9. Divide $2x^4y^3z$ by $-2x^3y$, and $-10b^2x^3z$ by bx.

$$\frac{2x^4y^3z}{-2x^3y}=-xy^2z.\ quot.\qquad \frac{-10b^2x^3z}{bx}=-10bx^2z.\ quot.$$

10. Divide $-x^4y^3z$ by $-x^2$, and $4x^3y^4z$ by y^4z. *Quot.* x^2y^3z *and* $4x^3z$.

48. *When the divisor will not divide the dividend exactly.*

RULE. In this case the quotient will have two terms, a numerator and a denominator, like a fraction [s].

I. Divide both coefficients by their greatest common measure, and each quotient will be the coefficient of its corresponding term, in the quotient required.

II. If the dividend and divisor contain the same letter, subtract the less index from the greater, and place the remainder

---

[r] Since the divisor, multiplied into the quotient, must produce the dividend, it evidently follows, that the divisor and quotient must, either jointly or severally, contain all the letters of the dividend; wherefore, if the divisor contain not all those letters, it is plain that the remaining ones must be found in the quotient; otherwise these two terms multiplied together would not produce the dividend as they ought.

[s] Division takes place only when the dividend contains the divisor; when that is not the case, actual division is evidently impossible, and all that can be done is, to set the dividend above a small line, and the divisor below it, to consider the whole as a fraction, and (by throwing out every thing common to both terms) reduce this fraction to its lowest terms, which will be the most convenient expression possible of the quotient required.

as an index over that letter, in the term of the quotient, which corresponds with the greater, whether it be the upper term, or lower.

III. Proceed in this manner with all the letters that are alike in both terms, and if there are unlike letters, they must be placed in the quotient, each in the term corresponding to that from which it was taken.

Like signs give + in the quotient, and unlike −, as before.

11. Divide $12\,a^2x^3y$ by $-15\,a^4x$.

OPERATION.

$$\frac{12\,a^2x^3y}{-15\,a^4x}=-\frac{4\,x^2y}{5\,a^2},\ quot.$$

Explanation.

Unlike signs give −, then the greatest common measure of 12 and 15 being 3, I divide both by it, and place the quotients 4 and 5 each in its proper term, after the sign −; next $4-2=2$ for the index of a in the *lower* term of the quotient, and $3-1=2$ for the index of x in the upper; then y being found alone in the upper, I place it in the upper term of the quotient.

12. Divide $-4\,x^5y^2z$ by $5\,x^2y^4$, and abc^2 by a^2bc.

$$\frac{-4\,x^5y^2z}{5\,x^2y^4}=-\frac{4\,x^3z}{5\,y^2}.\ quot.\quad \frac{abc^2}{a^2bc}=\frac{c}{a}.\ quot.$$

13. Divide $-12\,x^4yz$ by $-4\,x^3yz^4$, and $27\,a^3$ by $-54\,a^3x$.

$$\frac{-12\,x^4yz}{-4\,x^3yz^4}=\frac{3\,x}{z^3}.\ quot.\quad \frac{27\,a^3}{-54\,a^3x}=-\frac{1}{2x}.\ quot.$$

14. Divide x^2y by $4\,yz$, and $-20\,a^2z$ by $15\,az$. *Quot.* $\frac{x^2}{4z}$ *and* $-\frac{4\,a}{3}$.

15. Divide $10\,xy$ by $-30\,az$, and $2\,cx$ by $3\,az$. *Quot.* $-\frac{xy}{3az}$ *and* $\frac{2\,cx}{3\,az}$.

16. Divide $-ax$ by $-5\,ax$, and $-8\,ab^4$ by $4\,a^2b^2c$. *Quot.* $\frac{1}{5}$ *and* $-\frac{2\,b^2}{ac}$.

49. *When the dividend is a compound quantity.*

RULE. Divide every term of the dividend by the divisor, and connect the quotients together by their proper signs; the result will be the quotient required[t].

---

[t] The reason of this rule will be obvious from this consideration, namely, that the whole quotient is made up of all its constituent parts; now these parts

17. Divide $6x^2+12xy-9xyz$ by $3x$.

Explanation.

OPERATION.

$$\frac{6x^2+12xy-9xyz}{3x}=2x+4y-3yz.\ quot.$$

First $\frac{6x^2}{3x}=2x$

Secondly $\frac{12xy}{3x}=4y$

Thirdly $\frac{-9xyz}{3x}=-3yz$.

These three quotients, connected by their proper signs, give $2x+4y-3yz$ for the quotient required.

18. Divide $3a^3b-4ab^3$ by $5ax$.

OPERATION.

$$\frac{3a^3b-4ab^3}{5ax}=\frac{3a^2b}{5x}-\frac{4b^3}{5x}=\frac{3a^2b-4b^3}{5x}.\ quot.$$

Explanation.

First $\frac{3a^3b}{5ax}=\frac{3a^2b}{5x}$, secondly $\frac{-4ab^3}{5ax}=-\frac{4b^3}{5x}$; these two quotients having a common denominator $5x$, both numerators are placed over it, thus $\frac{3a^2b-4b^3}{5x}$, for the quotient.

19. Divide $21xyz-14ab$ by $-7xy$.

$$\frac{21xyz-14ab}{-7xy}=-3z+\frac{2ab}{xy}.\ quot.$$

20. Divide $4ab+6ax$ by $2a$. *Quot.* $2b+3x$.

21. Divide $9a^2-3ax-6$ by $-3a$. *Quot.* $-3a+x+\frac{2}{a}$.

22. Divide $x^2-2xy+4y^2$ by $4xy$. *Quot.* $\frac{x}{4y}-\frac{1}{2}+\frac{y}{x}$.

23. Divide $10xy-20x-y$ by $-5x$. *Quot.* $-2y+4+\frac{y}{5x}$.

50. *When the terms are both compound quantities.*

RULE I. Place the divisor to the left of the dividend, and range the terms of both in the same order, so that the highest power of some letter (viz. the same in both) may stand first, the next highest second, and so on.

---

must evidently arise from the division of each member separately of the dividend, by the divisor; and all the parts taken together (connected by their proper signs) will therefore constitute the quotient, according to the rule.

II. Divide the *first term* of the dividend by the *first term* of the divisor, by the preceding rules, and place the result with its proper sign in the quotient.

III. Multiply *the whole divisor* by this result, and place the products under their like in the dividend.

IV. Subtract these products from the quantities under which they stand, and to the remainder bring down one of the terms (or more, if necessary) of the dividend which has not been used.

V. Divide the first term of this last line by the first of the divisor, place the result with its proper sign in the quotient, multiply the whole divisor by this result, place the products under their like in the said last line, subtract, bring down, &c. as before, until all the terms of the dividend are brought down, and the work will be finished.

VI. If there be any remainder at last, place it over the divisor like a fraction, and subjoin this fraction with its proper sign to the quotient *.

---

* This rule is evident from what has been said in the preceding note. The mode of operation is similar to that employed in arithmetical Long Division, and will be easily understood by those who are expert in that rule. The examples should be all proved, both by multiplication and addition. The multiplication proof consists in multiplying the quotient and divisor together, and adding the remainder (if any) to the product, which will be like the dividend, if the work is right. The proof by addition consists in adding all the lower lines (viz. the second line in each compartment) of the operation and the remainder together, and if the operation be right, the sum will be like the dividend.

Proof of Ex. 24.

Mult. $a + b$ *the quotient.*
By $a + b$ *the divisor.*

$$\begin{array}{l} a^2 + ab \\ \phantom{a^2 +{}} ab + b^2 \\ \hline \end{array}$$

Produces $a^2 + 2ab + b^2$ *the dividend.*

Proof of Ex. 25.

Mult. $x^2 - 2xz + z^2$ *quot.*
By $x - z$ .... *divisor.*

$$\begin{array}{l} x^3 - 2x^2z + xz^2 \\ - x^2z + 2xz^2 + z^3 \\ \hline \end{array}$$

Produces $x^3 - 3x^2z + 3xz^2 + z^3$ *divid.*

24. Divide $a^2+2ab+b^2$ by $a+b$.

OPERATION.

$$\begin{array}{l} a+b)\ a^2+2ab+b^2\ (a+b \\ \quad\ \ \underline{a^2+\ ab\quad\quad} \\ \quad\quad\quad\ ab+b^2 \\ \quad\quad\quad\ \underline{ab+b^2} \\ \quad\quad\quad\ \ *\quad * \end{array}$$

Explanation.

I first divide a^2 by a, and place the result a in the quotient; I then multiply *the whole divisor* by a, and place the products a^2+ab under their like. I then subtract them from the quantities above them, and to the remainder ab, bring down $+b^2$, making $ab+b^2$; I divide the first of this, viz. ab, by the first of the divisor, viz. a, place the result b with its proper sign $+$ in the quotient, and then multiply the whole divisor $a+b$ by it, making $ab+b^2$, which I place under the last line.

25. Divide $x^3-3x^2z+3xz^2-z^3$ by $x-z$.

OPERATION. *Explanation.*

$$\begin{array}{lll} x-z)x^3-3x^2z+3xz^2-z^3(x^2-2xz+z^2 & = & \textit{quotient.} \\ \quad\quad\ \underline{x^3-\ x^2z}\ \ldots\ldots\ldots\ldots & = & \overline{x-z}\times x^2. \\ \quad\quad\quad -2x^2z+3xz^2\ \ldots\ldots\ldots & = & \textit{the rem. with } 3xz^2 \textit{ brought [down.} \\ \quad\quad\quad \underline{-2x^2z+2xz^2}\ \ldots\ldots\ldots & = & \overline{x-z}\times -2xz. \\ \quad\quad\quad\quad\quad\quad xz^2-z^3\ \ldots\ldots & = & \textit{the rem. with} -z^3 \textit{ brought [down.} \\ \quad\quad\quad\quad\quad\quad \underline{xz^2-z^3}\ \ldots\ldots & = & \overline{x-z}\times z^2. \\ \quad\quad\quad\quad\quad\quad\ \ *\quad * & & \\ \quad\quad \overline{\underline{x^3-3x^2z+3xz^2-z^3}} & & \textit{the proof by addition.} \end{array}$$

Divide the following quantities.

26.

$$\begin{array}{l} a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2, \textit{ quotient.} \\ \quad\quad\ \underline{a^3+\ a^2x\quad\quad\quad} \\ \quad\quad\quad\quad 4a^2x+5ax^2 \\ \quad\quad\quad\quad \underline{4a^2x+4ax^2} \\ \quad\quad\quad\quad\quad\quad\ ax^2+x^3 \\ \quad\quad\quad\quad\quad\quad\ \underline{ax^2+x^3} \\ \quad\quad\quad\quad\quad\quad\quad *\quad * \end{array}$$

27.

$$\begin{array}{l} z-3)z^3-6z-9(z^2+3z+3, \textit{ quotient.} \\ \quad\quad\ \underline{z^3-3z^2\quad} \\ \quad\quad\quad\quad 3z^2-6z \\ \quad\quad\quad\quad \underline{3z^2-9z} \\ \quad\quad\quad\quad\quad\quad 3z-9 \\ \quad\quad\quad\quad\quad\quad \underline{3z-9} \\ \quad\quad\quad\quad\quad\quad\ *\quad * \end{array}$$

28.

$$1-x)1 \quad (1+x+x^2+x^3+\frac{x^4}{1-x} \text{ quotient with the remainder.}$$
$$\underline{1-x}$$
$$x$$
$$\underline{x-x^2}$$
$$x^2$$
$$\underline{x^2-x^3}$$
$$x^3$$
$$\underline{x^3-x^4}$$
$$x^4 \text{ remainder.}$$

29 [*].

$$a^3-a^2x-ax^2+x^3)a^4-x^4 \quad (\quad a+x \text{ quotient.}$$
$$\underline{a^4-a^3x-\ a^2x^2+ax^3}$$
$$a^3x+\ a^2x^2-ax^3-\ x^4$$
$$\underline{a^3x-\ a^2x^2-ax^3+\ x^4}$$
$$2a^2x^2 \quad * \quad -2x^4 \text{ remainder.}$$

The quotient, with the remainder subjoined, is therefore $a+x+\dfrac{2\,a^2x^2-2x^4}{a^3-\ a^2x\ -ax^2+x^3}$.

30. Divide $x^2+2xy+y^2$ by $x+y$. *Quot.* $x+y$.
31. Divide $a^2-2\,ab+b^2$ by $a-b$. *Quot.* $a-b$.
32. Divide $ac+bc+ad+bd$ by $a+b$. *Quot.* $c+d$.
33. Divide $z^3-10\,z^2+33z-36$ by $z-4$. *Quot.* $z^2-6\,z+9$.
34. Divide $96-6\,a^4$ by $6-3\,a$. *Quot.* $16+8\,a+4\,a^2+2\,a^3$.
35. Divide z^3-1 by $z-1$. *Quot.* z^2+z+1.
36. Divide $x^3-9\,x^2+28x-29$ by $x^2-6\,x+9$. *Quot.* $x-3+\dfrac{x\ -2}{x^2-6x+9}$.

---

[*] In ex. 29. the $-x^4$ is not brought down until it is wanted in the last step. The following example shews how to manage when there are compound factors in the work.

$$x-z)\ x^3-ax^2+bx-c\ (x^2+\overline{z-a}.x+\overline{z^2-az+b}$$
$$\underline{x^3-zx^2}$$
$$+\overline{z-a}.x^2+bx$$
$$\underline{+\overline{z-a}.x^2-\overline{z^2-az}.x}$$
$$+\overline{z^2-az+b}.x-c$$
$$\underline{+\overline{z^2-az+b}.x-z^3+az^2-bz}$$
$$\textit{Remainder}\ +z^3-az^2+bz-c$$

FRACTIONS.

51. *To reduce an algebraic fraction, both terms of which are compound quantities, to its lowest terms.*

RULE I. Find the greatest common measure of both terms of the fraction, as in arithmetical fractions, observing at every step before you divide, to throw out from the divisor the greatest simple quantity which is common to each of its terms, using the remaining part instead of the whole: the last divisor is the greatest common measure.

II. Divide both terms of the given fraction by the greatest common measure, and the quotient will be the respective terms of the new fraction, which will be the given one in its lowest terms [y].

---

[y] This rule depends on the following principles, namely, First. If one quantity measure another, it will measure any multiple of that quantity.

Secondly. If a quantity measure two others, it will measure their sum and difference.

Thirdly. If one quantity be divided by another, and the preceding divisor by the remainder, and so on continually, the remainder will at length be less than any given quantity.

Wherefore (example 1.) since $x-z$ measures x^2-z^2, it will likewise measure $x \times \overline{x^2-z^2} + z^2 \times \overline{x-z}$.

But this quantity is $=x^3-z^3$, as appears by actually multiplying and adding the terms, according to the import of the signs; wherefore $x-z$ measures both x^2-z^2 and x^3-z^3, namely, both terms of the given fraction.

But $x-z$ is likewise the greatest quantity that will measure both terms; for if not, let a greater common measure be assumed; then since this greater measures x^2-z^2 and x^3-z^3, it will likewise measure their difference, namely, xz^2-z^3.

Now all the compound divisors of xz^2-z^3 besides $x-z$, are xz^2-z^3 and $xz-z^2$, neither of which, as appears by trial, will measure x^2-z^2, wherefore $x-z$ is the greatest common measure; the rule therefore is manifest.

The throwing out the greatest simple quantity common to every term of the divisor, while it simplifies the operation, does not in the least interfere with the compound divisor we are in search of.

EXAMPLES.

1. Reduce $\dfrac{x^3-z^2}{x^3-z^3}$ to its lowest terms.

OPERATION.

$$
\begin{array}{l}
x^3-z^2)x^3-\;z^3\;(x \\
\qquad\qquad x^3-xz^2 \\
\hline
\end{array}
$$

Throw out z^2, *and* xz^2-z^3

becomes . . . . . $x-z\,)x^2-z^2\,(x+z$

$$
\begin{array}{r}
x^2-xz \\
\hline
xz-z^2 \\
xz-z^2 \\
\hline
\end{array}
$$

Wherefore $x-z$ *is the greatest common measure, now divide both terms of the given fraction by it: and first the numerator, thus,* $x-z)x^2-z^2(x+z$ *(as before,) for the numerator; then the denominator, thus,*

$$
\begin{array}{l}
x-z)x^3-z^3\;(x^2+xz+z^2\ \textit{denominator.} \\
\quad x^3-x^2z \\
\hline
\qquad x^2z \\
\qquad x^2z-xz^2 \\
\hline
\qquad\quad xz^2-z^3 \\
\qquad\quad xz^2-z^3 \\
\hline
\end{array}
$$

Therefore $\dfrac{x+z}{x^2+xz+z^2}$ *is the given fraction in its lowest terms.*

Explanation.

I divide the denominator by the numerator, and from the remainder $xz-z^3$ throw out z^2, which is common to both terms, and $x-z$ is left: with this I divide the divisor x^2-z^2, and as the operation leaves no remainder, $x-z$ is the greatest common measure.

I next divide both terms of the given fraction by $x-z$, and the quotients $x+z$, and x^2+xz+z^2, are the terms of the answer, or the lowest terms required.

2. Reduce $\dfrac{a^4-b^4}{a^3-a^2b-ab^2+b^3}$ to its lowest terms.

Thus, $a^3-a^2b-ab^2+b^3)a^4-b^4\quad *\quad *\;(a+b$

$$
\begin{array}{r}
a^4-a^3b-\;a^2b^2+ab^3 \\
\hline
a^3b+\;a^2b^2-ab^3-\;b^4 \\
a^3b-\;a^2b^2-ab^3+\;b^4 \\
\hline
2a^2b^2\quad *\quad -2b^4
\end{array}
$$

and leaving out $2b^2$, *which occurs in each term of this remainder, we have* a^2-b^2 *for the next divisor; then,*

$$
\begin{array}{l}
a^2-b^2)a^3-a^2b-ab^2+b^3\,(a-b \\
\qquad\quad a^3\quad *\ -ab^2 \\
\hline
\qquad\quad a^2b\quad *\ -b^3 \\
\qquad\quad a^2b\quad *\ -b^3 \\
\hline
\end{array}
$$

Whence a^2-b^2 *is the greatest common measure: and dividing both terms of the given fraction by it, we shall have*

$\frac{a^2+b^2}{a-b}$, *the lowest terms required.*

3. Reduce $\frac{a^3-az^2}{a^2+2az+z^2}$ to its lowest terms.

Thus $a^2+2az+z^2)a^3-az^2 \ldots\ldots (a-2z$

$$
\begin{array}{r}
a^3+2a^2z-az^2 \\
\hline
-2a^2z-2az^2 \\
-2a^2z-4az^2-2z^3 \\
\hline
+2az^2+2z^3
\end{array}
$$

this, by dividing by the common quantity $2z^2$, *becomes* $a+z$.

Then $a+z)a^2+2az+z^2(a+z$

$$
\begin{array}{r}
a^2+az \\
\hline
az+z^2 \\
az+z^2 \\
\hline
\end{array}
$$

Whence $a+z$ *is the common measure.*

And dividing both terms of the given fraction by it, $\frac{a^2-az}{a+z}$ *will be the lowest terms required.*

4. Reduce $\frac{a^5-a^3x^2}{a^4-x^4}$ to its lowest terms. *Ans.* $\frac{a^3}{a^2+x^2}$.

5. Reduce $\frac{a+z}{a^2+2az+z^2}$ to its lowest terms. *Ans.* $\frac{1}{a+z}$.

6. Reduce $\frac{a^2y-y}{ab-b}$ to its lowest terms. *Ans.* $\frac{ay-y}{b}$.

The remaining operations in Arithmetical and Algebraic fractions being exactly alike, it was not judged necessary to repeat the rules, but merely to give an example or two under each, for the learner's exercise.

7. Reduce $ax-\frac{x}{y}$ and $z+\frac{a}{a+b}$ to equivalent improper fractions. (For the rule, see Art. 172. Part I.)

Thus $ax-\frac{x}{y}=\frac{axy-x}{y}$ *Ans.* *And* $z+\frac{a}{a+b}=\frac{az+bz+a}{a+b}$ *Ans.*

8. Reduce $a+\frac{b}{c}$ and $x+y-\frac{z}{a}$ to improper fractions. *Answer* $\frac{ac+b}{c}$ *and* $\frac{ax+ay-z}{a}$.

9. Reduce $\frac{3\,xy+5\,z}{x}$ and $\frac{4\,ab-8\,bc-3\,x}{4\,b}$ to mixed quantities. (See Art. 173. Part I.)

Thus $\frac{3\,xy+5\,z}{x}=3\,y+\frac{5\,z}{x}$ *Ans.* *And* $\frac{4\,ab-8\,bc-3\,x}{4\,b}=a-2\,c-\frac{3\,x}{4\,b}$ *Ans.*

10. Reduce $\frac{9\,x^2y+2\,y^2}{3\,x}$ and $\frac{x^2-2\,xy+3\,y^2}{x-y}$ to equal mixed quantities. *Ans.* $3\,xy+\frac{2\,y^2}{3\,x}$ and $x-y+\frac{2\,y^2}{x-y}$.

11. Reduce $\frac{x+\frac{x}{y}}{z}$ and $\frac{2\,y}{3\,x-\frac{a}{b}}$ to simple fractions. (Art. 177. Part I.)

Thus $\frac{x+\frac{x}{y}}{z}=\frac{xy+x}{zy}$ *Ans.* *And* $\frac{2\,y}{3\,x-\frac{a}{b}}=\frac{2\,by}{3\,bx-a}$ *Ans.*

12. Reduce $\frac{x+\frac{y}{z}}{y-\frac{x}{y}}$ and $\frac{3\,xy-\frac{2\,z}{3}}{4\,z\;+\frac{3\,x}{4\,y}}$ to simple fractions. (Art. 178. Part I.) *Ans.* $\frac{xyz+y^2}{y^2z-xz}$ *and* $\frac{36\,xy^2-8\,yz}{48\,yz\;+9\,x}$.

13. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$, to equivalent fractions, having a common denominator. (Art. 180. Part I.)

Thus $\left.\begin{array}{l} a\times d\times y=ady \\ c\times b\times y=cby \\ x\times b\times d=bdx \end{array}\right\}$ *new numerators.*

$b\times d\times y=bdy$, *common denominator.*

Ans. $\frac{ady}{bdy}$, $\frac{cby}{bdy}$, *and* $\frac{bdx}{bdy}$.

14. Reduce $\frac{a}{2\,x}$, $\frac{x}{y}$, and $\frac{x-8}{z^2}$, to a common denominator.

Ans. $\frac{ayz^2}{2\,xyz^2}$, $\frac{2\,x^2z^2}{2\,xyz^2}$, *and* $\frac{2\,x^2y-16\,xy}{2\,xyz^2}$.

15. Required the sum and difference of $\frac{a}{b}$ and $\frac{c}{d}$? (Art. 187. 192. Part I.)

Thus $\left.\begin{array}{l} a\times d=ad \\ c\times b=bc \end{array}\right\}$ *new numerators.*

$b\times d=bd$, *common denominator.*

Then $\frac{ad+bc}{bd}=$ *the sum,* $\frac{ad-bc}{bd}=$ *the difference.*

16. Required the sum and difference of $\frac{4y}{5z}$ and $\frac{2y}{3x}$?

Sum $\frac{12xy+10yz}{15xz}$. *Diff.* $\frac{12xy-10yz}{15xz}$.

17. Required the sum and difference of $\frac{4a}{5}$ and $\frac{3a}{7}$? *Sum* $\frac{43a}{35}$. *Diff.* $\frac{13a}{35}$[*].

18. Multiply $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$, together. (Art. 198. Part I.)

Thus $\frac{a}{b}\times\frac{c}{d}\times\frac{x}{y}=\frac{acx}{bdy}$ *the product.*

19. Multiply $\frac{2x}{y}$, $\frac{3x}{2z}$, and $\frac{5yz}{3x}$, together. *Prod.* $5x$[a].

20. Divide $\frac{a}{2bc}$ by $\frac{3ay}{4cd}$. (Art. 204. Part I.)

Thus $\frac{a}{2bc}\times\frac{4cd}{3ay}=\frac{2d}{3by}$ *quotient.*

---

[*] In like manner the sum of $\frac{1}{a+b}$ and $\frac{1}{a-b}$ will be $\frac{a-b}{a^2-b^2}+\frac{a+b}{a^2-b^2}=\frac{a-b+a+b}{a^2-b^2}=\frac{2a}{a^2-b^2}$, and their difference, or $\frac{1}{a+b}-\frac{1}{a-b}=\frac{a-b}{a^2-b^2}-\frac{a+b}{a^2-b^2}=\frac{a-b-a-b}{a^2-b^2}=-\frac{2b}{a^2-b^2}$; also $\frac{x+y}{x-y}\pm\frac{x-y}{x+y}=\frac{x^2+2xy+y^2\pm x^2\mp 2xy\pm xy}{x^2-y^2}=\frac{2x^2+2y^2}{x^2-y^2}$ for the sum, and $\frac{4xy}{x^2-y^2}$ for the difference.

[a] Thus also the product of $\frac{x+z}{x-z}\times\frac{x+z}{x-z}=\frac{x^2+2xz+z^2}{x^2-2xz+z^2}$, and the quotient, or $\frac{x+z}{x-z}\times\frac{x-z}{x+z}=\frac{x^2-z^2}{x^2-z^2}=1$.

21. Divide $\frac{4x}{b}$ by $\frac{4x}{3a}$, and $\frac{x}{z}$ by $\frac{a}{b}$. *Quot.* $\frac{3a}{b}$ *and* $\frac{bx}{az}$.

22. Divide $\frac{xy}{3}$ by $\frac{8y}{9}$, and $\frac{xy}{4z}$ by $\frac{xy}{3}$. *Quot.* $\frac{3x}{8}$ *and* $\frac{3}{4z}$.

23. Add the sum, difference, product, and quotient, of $\frac{3a}{4}$ and $\frac{2a}{5}$ together. *Ans.* $\frac{12a^2+60a+75}{40}$.

INVOLUTION[b].

52. *To involve simple quantities to any power.*

RULE I. Involve the coefficient to the power required, for a coefficient.

II. Multiply the index of each letter by the index of the required power.

---

[b] Involution being nothing more than continued multiplication, it will be sufficient to refer the reader to the notes on multiplication, where he will find the reason of this rule.

The Arabians denominated the powers from the consideration of the *products* of the indices, calling them the *square, cube, biquadrate, sursolid, cube-squared, second-sursolid, quadrato-quadrato-quadratum, cube-of-the-cube, square-of-the-sursolid, third-sursolid*, &c.

Diophantus, Vieta, Oughtred, and others, named the powers according to the *sums* of the indices, as *root, square, cube, quadrato-quadratum, quadrato-cubus, cubo-cubus, quadrato-quadrato-cubus, quadrato-cubo-cubus, cubo-cubo-cubus*, &c.

Des Cartes, and most of the writers since his time, employ a much simpler method than either of the former, calling the powers respectively, the 1st, 2nd, 3rd, 4th, &c. according to the index. And because the side of a square multiplied into itself gives the area of the square, and the side of a cube multiplied continually twice into itself produces the solidity of the cube, the terms *square* and *cube* have been applied to numbers arising from like operations. Hence it is that the product of a number multiplied into itself is called a square; and if the number be multiplied twice into itself, the product is called a cube.

It is evident that all the powers of an affirmative quantity will be affirmative, for + into + always produces +; likewise that all the even powers of a negative quantity will be +, and the odd powers —; for since — into — produces +, and this into — produces —, and this into — produces +, and so on alternately + and —; it follows that the even powers will be +, and the odd powers —.

III. Place each product over its respective letter, and prefix the coefficient found above, the result will be the power required.

IV. Of an affirmative quantity all the powers will be $+$; and of a negative quantity the odd powers will be $-$, and the even powers $+$.

EXAMPLES.

1. Involve $2a^2$ to the third power.

OPERATION.

$\overline{2}|^3=2\times2\times2=8$ *the coefficient.*

$a^{2\times3}=a^6$ *the literal part.*

Wherefore $\overline{2a^2}|^3=8a^6$, *answer.*

Explanation.

I first find the coefficient, by multiplying 2 twice into itself; then I multiply 2, the index of a, into 3, the index of the third power; lastly I prefix the coefficient 8 to a^6, making $8a^6$ for the power required.

2. Involve $3ax^2$ to the fourth power.

Thus $\overline{3ax^2}|^4=81a^4x^8$, *answer.*

3. Involve $-2z$ to the square, and $-3y^2$ to the cube. $\overline{-2z}|^2=4z^2$ *the square, and* $\overline{-3y^2}|^3=-27y^6$ *the cube.*

4. Involve $-\frac{2z}{3y^2}$ to the second and third powers. *Ans. second power* $\frac{4z^2}{9y^4}$, *third power* $-\frac{8z^3}{27y^6}$.

5. Involve xy^2z^3 to the third power. *Ans.* $x^3y^6z^9$.

6. Involve $-\frac{ax^2}{3z}$ to the third power. *Ans.* $-\frac{a^3x^6}{27z^3}$.

53. *To involve compound quantities.*

RULE. Multiply the given quantity as many times continually into itself wanting one, as there are units in the index of the required power, and the last product will be the power required[e].

---

[e] This rule is merely multiplication, and depends on the same principles.

7. **Involve $a+x$ and $a-x$, each to the cube.**

OPERATION.

$$
\begin{array}{lll}
a+x & \textit{first power.} & a-x \\
\underline{a+x} & & \underline{a-x} \\
a^2+ax & & a^2-ax \\
\underline{\quad\quad ax+x^2} & & \underline{\quad -ax+x^2} \\
a^2+2ax+x^2 & \textit{second power.} & a^2-2ax+x^2 \\
\underline{a+x} & & \underline{a-x} \\
a^3+2a^2x+ax^2 & & a^3-2a^2x+ax^2 \\
\underline{\quad\quad a^2x+2ax^2+x^3} & & \underline{\quad -a^2x+2ax^2-x^3} \\
a^3+3a^2x+3ax^2+x^3 & \textit{cube.} & a^3-3a^2x+3ax^2-x^3
\end{array}
$$

Explanation.

I multiply $a+x$ and $a-x$ each by itself, and the product is the square; I multiply this by $a+x$ in one, and $a-x$ in the other, and the product in each is the cube: it will be seen that these operations differ only in the signs of their *even* terms.

8. **Involve $x\pm z$ to the fifth power.**

OPERATION.

$$
\begin{array}{ll}
x\pm z & \textit{first power.} \\
\underline{x\pm z} & \\
x^2\pm xz & \\
\underline{\quad \pm xz+z^2} & \\
x^2\pm 2xz+z^2 & \textit{second power.} \\
\underline{x\pm z} & \\
x^3\pm 2x^2z+xz^2 & \\
\underline{\quad \pm x^2z+2xz^2\pm z^3} & \\
x^3\pm 3x^2z+3xz^2\pm z^3 & \textit{third power.} \\
\underline{x\pm z} & \\
x^4\pm 3x^3z+3x^2z^2\pm xz^3 & \\
\underline{\quad \pm x^3z+3x^2z^2\pm 3xz^3+z^4} & \\
x^4\pm 4x^3z+6x^2z^2\pm 4xz^3+z^4 & \textit{fourth power.} \\
\underline{x\pm z} & \\
x^5\pm 4x^4z+6x^3z^2\pm 4x^2z^3+xz^4 & \\
\underline{\quad \pm x^4z+4x^3z^2\pm 6x^2z^3+4xz^4\pm z^5} & \\
x^5\pm 5x^4z+10x^3z^2\pm 10x^2z^3+5xz^4\pm z^5 & \textit{fifth power.}
\end{array}
$$

Explanation.

The only difficulty in this example is how to work with the double sign $\pm$; First, $x\times x=x^2$, secondly, $x\times\pm z=\pm xz$, which is the first line; then $\pm z\times x=\pm xz$, and $\pm z\times\pm z=+z^2$, which is the second line of the work; also $\pm xz$ added to $\pm xz$, gives $\pm 2xz$ in the next line, and so on throughout.

9. **Involve $x+1$ to the cube.** *Ans.* x^3+3x^2+3x+1.

10. **Involve $2x-3y$ to the cube.** *Ans.* $8x^3-36x^2y+54xy^2-27y^3$.

11. **Involve $3-x$ to the fourth power.** *Ans.* $81-108x+54x^2-12x^3+x^4$.

12. Involve $a+b-c$ to the square. *Ans.* $a^2+2ab-2ac+b^2-2bc+c^2$.

SIR ISAAC NEWTON'S RULE FOR INVOLUTION[d].

Whereby any power of a given compound quantity may be obtained by an easy and expeditious mental operation.

54. *For Binomials.*

To find the terms and indices.

RULE I. Write down the *leading* quantity successively, as many times as there are units in the index of the required power.

II. Over the first of these place the index of the power; over the second, the index decreased by 1; over the third, the index decreased by 2; and so on, making the index of each term always 1 less than that of the preceding term.

III. Subjoin the *following* quantity to the second, and every succeeding term, of the above, and carry it one place beyond.

IV. Make 1 (understood) the index of the first of these; 2 the index of the second; 3 of the third, and so on, constantly increasing by 1 to the last; the index of which will be that of the required power.

Thus, if it be required to involve $a-z$ to the fifth power, the quantities and indices will stand as follows.

The leading quantity a, thus,	a^5,	a^4,	a^3,	a^2,	a.	
The following quantity z		z,	z^2,	z^3,	z^4,	z^5.
Both quantities connected ...	a^5,	a^4z,	a^3z^2,	a^2z^3,	az^4,	z^5.

To find the coefficients of the terms.

RULE I. The coefficient of the first term is always 1, (understood,) and that of the second term is always the index of the required power.

---

[d] This rule is a branch of the celebrated Binomial Theorem, discovered by Sir Isaac Newton in 1669. The author first discovered it by induction, namely, by observing the law which the signs, coefficients, and indices invariably follow, in a Binomial actually involved to several different powers. It will be a profitable employ for the learner to make the same induction: let him compare the 8th example with Newton's rule, and he will see that the coefficients, indices, signs, &c. of the terms, *in every one of the powers,* observe invariably the law on which the rule is founded.

II. Multiply the coefficient and index of the leading quantity in the second term together, divide the product by 2, and the quotient will be the coefficient of the third term.

III. And in general, if the coefficient and index of the leading quantity in any term be multiplied together, and the product divided by the number denoting the place of that term, the quotient will be the coefficient of the next succeeding term.

Thus, in the above example.

The coefficient of the first term will be 1, *understood.*

That of the second term .... 5, *or the index of the power.*

That of the third .... $\frac{5\times4}{2}=10.$

That of the fourth ...... $\frac{10\times3}{3}=10.$

That of the fifth .... $\frac{10\times2}{4}=5.$

And that of the sixth .... $\frac{5\times1}{5}=1$, *understood.*

Thus, the terms connected with their coefficients in the above example, will be x^5, $5x^4z$, $10x^3z^2$, $10x^2z^3$, $5xz^4$, z^5.

To find the signs.

When the signs of both terms of the Binomial are +, all the signs of the power will be +; but when the second term is —, all the odd terms (the 1st, 3rd, 5th, &c.) will be +, and the even terms (the 2nd, 4th, 6th, &c.) —. *Thus in the example above proposed,* where it is required to involve $a-z$ to the fifth power; the second term being —, the 1st, 3rd, and 5th terms of the power will be +, and the 2nd, 4th, and 6th terms —.

Wherefore $x^5-5x^4z+10x^3z^2-10x^2z^3+5xz^4-z^5$ will be the fifth power (signs, coefficients, and indices complete) of the given quantity $x-z$.

13. Involve $a+b$ to the sixth power, by Newton's rule.

The leading quantity a... a^6, a^5, a^4, a^3, a^2, a.

The following quantity b..... b, b^2, b^3, b^4, b^5, b^6.

Both connected....... a^6, a^5b, a^4b^2, a^3b^3, a^2b^4, ab^5, b^6.

The coefficients 1, 6, $\frac{6\times5}{2}$, $\frac{15\times4}{3}$, $\frac{20\times3}{4}$, $\frac{15\times2}{5}$, $\frac{6\times1}{6}$.

Or, 1, 6, 15, 20, 15, 6, 1.

The signs of the terms are all $+$.

Wherefore by connecting the signs, coefficients, and terms, $\overline{a+b}|^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 =$ *the power required.*

14. Involve $a-z$ to the 9th power.

Terms.

$a^9,\ a^8z,\ a^7z^2,\ a^6z^3,\ a^5z^4,\ a^4z^5,\ a^3z^6,\ a^2z^7,\ az^8,\ z^9.$

Coefficients.

$$1,\ 9,\ \frac{9\times 8}{2},\ \frac{36\times 7}{3},\ \frac{84\times 6}{4},\ \frac{126\times 5}{5},\ \frac{126\times 4}{6},\ \frac{84\times 3}{7},\ \frac{36\times 2}{8},\ \frac{9\times 1}{9}.$$

Or, 1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

The signs of the odd terms are $+$, *and the even terms* $-$.

Wherefore $\overline{a-z}|^9 = a^9-9a^8z+36a^7z^2-84a^6z^3+126a^5z^4-126a^4z^5+84a^3z^6-36a^2z^7+9az^8-z^9 =$ *the power required.*

15. Involve $x+3$ to the fourth power.

Terms $x^4,\ x^3\times 3,\ x^2\times 3^2,\ x\times 3^3,\ 3^4$.

Coefficients 1, 4, 6, 4, 1. *Signs all* $+$.

Wherefore $x^4+4x^3\times 3+6x^2\times 3^2+4x\times 3^3+3^4$, *that is*, $x^4+12x^3+54x^2+108x+81$, *is the power required.*

16. Involve $a+b$ to the square. *Ans.* $a^2+2ab+b^2$.

17. Find $\overline{x+y}|^3$. *Ans.* $x^3+3x^2y+3xy^2+y^3$.

18. Find $\overline{m-n}|^4$. *Ans.* $m^4-4m^3n+6m^2n^2-4mn^3+n^4$.

19. Find $\overline{a-z}|^5$. *Ans.* $a^5-5a^4z+10a^3z^2-10a^2z^3+5az^4-z^5$.

20. Find $\overline{z-1}|^7$. *Ans.* $z^7-7z^6+21z^5-35z^4+35z^3-21z^2+7z-1$.

21. Required the nth power of $a\pm x$? *Ans.* $a^{n} \pm na^{n-1}x + n.\frac{n-1}{2}a^{n-2}x^2 \pm n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}x^3 + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}a^{n-4}x^4 \pm$, &c. &c.

55. *For Trinomials**.

RULE. Let two of the terms of the given trinomial be consi-

* The ingenious Mr. Abraham Demoivre has given a method, whereby any power or root of a multinomial, consisting of any number of terms whatever, may be found, which may be seen in the Philosophical Transactions, No. 230.

dered as one factor, and the remaining term as the other, and proceed as before.

22. Involve $a+b+c$ to the third power.

Let a be one factor, and $b+c$ the other; then will $\overline{a+b+c}|^3 = a^3+3a^2.\overline{b+c}+3a.\overline{b+c}|^2+\overline{b+c}|^3$, *which by expanding the powers of $b+c$ becomes* $a^3+3a^2.\overline{b+c}+3a.\overline{b^2+2bc+c^2}+b^3+3b^2c+3bc^2+c^3$.

This by multiplication becomes $a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$, *the power required.*

23. Involve $x+y-z$ to the fourth power.

Let x be one factor, and $y-z$ the other; then will $\overline{x+y-z}|^4 = x^4+4x^3.\overline{y-z}+6x^2.\overline{y-z}|^2+4x.\overline{y-z}|^3+\overline{y-z}|^4$, *which expanded and multiplied, becomes* $x^4+4x^3y-4x^3z+6x^2y^2-12x^2yz+6x^2z^2+4xy^3-12xy^2z+12xyz^2-4xz^3+y^4-4y^3z+6y^2z^2-4yz^3+z^4$, *the fourth power required.*

Or thus.

Let $x+y$ be one factor, and $-z$ the other; then will $\overline{x+y-z}|^4 = \overline{x+y}|^4-4.\overline{x+y}|^3.z+6.\overline{x+y}|^2.z^2-4.\overline{x+y}.z^3+z^4$: *this expanded and multiplied, becomes* $x^4+4x^3y+6x^2y^2+4xy^3+y^4-4x^3z-12x^2yz-12xy^2z-4y^3z+6x^2z^2+12xyz^2+6y^2z^2-4xz^3-4yz^3+z^4$, *the fourth power as before.*

EVOLUTION.

56. *To extract the roots of simple quantities.*

Rule I. Place the fractional index denoting the required root, over the given quantity.

II. Extract the root of the coefficient, and the result will be the coefficient of the root.

III. Multiply the index of each letter by the index of the root, and place the results each over its proper letter for the literal part of the root, to which prefix the coefficient [f].

---

[f] The rule, as it respects the numeral coefficient, is sufficiently plain, being merely the evolution of the root of a number, which is explained in the Arithmetical part, (from Art. 268. to Art. 285.) With respect to the literal part, we may consider evolution and involution as parts of the same general rule, differing in one particular only, namely, the indices; those employed in invo-

Note. If the given quantity be *affirmative*, all its roots will be $+$; but if *negative*, its odd roots will be $-$, and its even ones impossible [g].

EXAMPLES.

1. Extract the square root of $9a^4$.

OPERATION.

$\overline{9a^4}|^{\frac{1}{2}}=3a^2$, *the root.*

Explanation.

Here $\frac{1}{2}$ is the index of the square root, and is placed over $9a^4$, then the square root of 9 is 3 for the coefficient, and $4\times\frac{1}{2}=2$ is the index; wherefore $3a^2$ is the root, the sign of which is $+$.

2. Extract the cube root of $-64x^3y^9$.
Thus $-\overline{64x^3y^9}|^{\frac{1}{3}}=-4xy^3$, *the root.*

3. Extract the fourth root of x^8y^4z.
Thus $\overline{x^8y^4z}|^{\frac{1}{4}}=x^2yz^{\frac{1}{4}}$, *the root.*

4. Extract the square root of $\frac{4x^2}{25z^4}$.

$$\text{Thus } \overline{\frac{4x^2}{25z^4}}\Big|^{\frac{1}{2}}=\frac{2x}{5z^2}, \text{ the root.}$$

5. Extract the square root of $4a^2x^4$. *Root* $2ax^2$.
6. Extract the cube root of $-8x^6z^3$. *Root* $-2x^2z$.
7. Extract the fourth root of $81y^8x^4z$. *Root* $3y^2xz^{\frac{1}{4}}$.
8. Extract the fifth root of $-x^5y^{10}$. *Root* $-xy^2$.
9. Extract the square root of $\frac{16a^4z}{49x^6y^2}$. *Root* $\frac{4a^2z^{\frac{1}{2}}}{7x^3y}$.
10. Find $-\overline{\frac{125x^6y^9}{343a^3}}\Big|^{\frac{1}{3}}$. *Root* $-\frac{5x^2y^3}{7a}$

57. *To extract the square root of compound quantities.*

RULE I. Range the quantities according to the dimensions of some letter concerned, as in division, Art. 50.

II. Find the root of the left hand term, (Art. 56.) set it in

---

lution being *integers*, and those in evolution *fractions*; hence the process in both cases is the same as far as it respects the letters, as may be seen by comparing this rule with the rule for involution.

[g] This is evident, because no quantity multiplied into itself can possibly produce a negative product; for $+$ into $+$ produces $+$, and $-$ into $-$ produces $+$; and therefore a negative power can have no *even* root, or in other words, *the even roots of all negative quantities are impossible.*

the quotient, and its square under the said left hand term, and subtract.

III. Bring down the two next terms to the remainder for a dividend, and set double the root on its left for a divisor.

IV. Divide the dividend by the divisor, and subjoin the result to both the quotient and the divisor.

V. Multiply the divisor (so increased) by the term last put in the quotient, subtract the product from the dividend, and bring down the two next terms to the remainder for a new dividend.

IV. Bring down the divisor, with its last term doubled, to the left of the new dividend, for a divisor; divide; subjoin the result to both quotient and divisor; multiply; subtract; bring down two terms, &c. proceeding in this manner until all the terms in the dividend are brought down, and the work is finished.

The operations may be proved either by addition or involution [b].

---

[b] Evolution being the converse of involution, necessarily requires a converse operation, which as far as it respects simple quantities is sufficiently plain; but when the roots of compound quantities are to be extracted, the mode of operating is not so obvious: indeed it does not appear that the rule here given is capable of an investigation *a priori*, but that it owes its origin to the mechanical process of trial or experiment. Thus it was known by involution, that the square of $a+b$ is $a^2+2ab+b^2$, but the difficulty was, how to frame a rule whereby the square being given the root could be obtained from it: the first member a of the root evidently arises by taking the root of a^2, the first member of the square; but the next inquiry was, how the other member b of the root could be had from $2ab+b^2$ (the remaining members of the square) without remainder. By trials it was found, that the first member a of the root being doubled, and the second member b added to it, the whole will form a divisor of $2ab+b^2$, which will exactly give b (without remainder) in the quotient; and this method (continued according to the precepts given above) will give the roots of all other compound quantities, as is proved by the converse process of involution, from whence alone the truth of the rule is manifest. In the same manner rules for finding the roots of higher powers were discovered, which, as they are not wanted in this place, are reserved for the exercise of the rules of approximation, given in the following part of the work.

11. **To extract the square root of $x^4+4x^3+6x^2+4x+1$.**

OPERATION.

$x^4+4x^3+6x^2+4x+1(x^2+2x+1=$ *the root.*

x^4 .... $=$ *the square of* x^2.

$2x^2+2x)\ldots 4x^3+6x^2$ .... $=$ *first dividend.*

$4x^3+4x^2$ .... $=(2x^2+2x)\times 2x.$

$2x^2+4x+1)\ldots\ldots 2x^2+4x+1$ .... $=$ *second dividend.*

$2x^2+4x+1$ .... $=(2x^2+4x+1)$ [$\times 1$.

$x^4+4x^3+6x^2+4x+1$ *proof by addition.*

Proof by involution.

x^2+2x+1 .... *root.*

x^2+2x+1 .... *root.*

$x^4+2x^3+x^2$

$2x^3+4x^2+2x$

x^2+2x+1

$x^4+4x^3+6x^2+4x+1$ *square.*

Explanation.

The greatest square in x^4 is x^4; I put its root x^2 in the quotient, subtract the square, and bring down $4x^3+6x^2$; I next double the root x^2, making $2x^2$, which I place on the left for a divisor; I divide $4x^3$ by $2x^2$, and place the result $2x$ both in the quotient and divisor, making the latter $2x^2+2x$; this I multiply by $2x$, making $4x^3+4x^2$, which, subtracted from the quantities above, leaves $2x^2$; to this I bring down $4x+1$, making $2x^2+4x+1$ for a new dividend; to the left of this I bring down the divisor, after doubling its last term, by which it becomes $2x^2+4x$; I divide the new dividend by this, placing the quotient figure 1 both in the quotient and divisor: then I multiply the divisor by it, place the product below the dividend, and the work is finished. The proof by addition, as well as that by involution, will be sufficiently obvious; the former being the sum of the several products, or *lower lines*, and the latter the root involved to the square.

12. Required the square root of $4x^4-12x^3+25x^2-24x+16$?

$4x^4-12x^3+25x^2-24x+16(2x^2-3x+4$ *root.*

$4x^4$

$4x^2-3x)\ldots -12x^3+25x^2$

$-12x^3+9x^2$

$4x^2-6x+4)$ .... $16x^2-24x+16$

$16x^2-24x+16$

$4x^4-12x^3+25x^2+24x+16$ *proof.*

13. Extract the square root of $a^2-ay+\frac{y^2}{4}$.

$$a^2-ay+\frac{y^2}{4}\left(a-\frac{y}{2}\right. \textit{ the root.}$$

$$a^2$$

$$a-\frac{y}{2}\Big)\quad -ay+\frac{y^2}{4}$$

$$-ay+\frac{y^2}{4}$$

14. Required the square root of $x^4+x^3y+\frac{x^2y^2}{4}-\frac{3x^2}{2}-\frac{3xy}{4}+\frac{9}{16}$?

$$x^4+x^3y+\frac{x^2y^2}{4}-\frac{3x^2}{2}-\frac{3xy}{4}+\frac{9}{16}\left(x^2+\frac{xy}{2}-\frac{3}{4}\right.$$

$$x^4$$

$$2x^2+\frac{xy}{2}\Big)\quad x^3y+\frac{x^2y^2}{4}$$

$$x^3y+\frac{x^2y^2}{4}$$

$$2x^2+xy-\frac{3}{4}\Big)\ \ldots\ldots\ldots -\frac{3x^2}{2}-\frac{3xy}{4}+\frac{9}{16}$$

$$-\frac{3x^2}{2}-\frac{3xy}{4}+\frac{9}{16}$$

15. Required the square root of x^2+2x+1? *Root* $x+1$.

16. Extract the square root of $x^2-2xy+y^2$. *Root* $x-y$.

17. Extract the square root of $16z^2-40z+25$. *Root* $4z-5$.

18. Find the square root of $y^4-6y^3+13y^2-12y+4$. *Root* y^2-3y+2.

19. Required the square root of $x^4-4x^3y+6x^2y^2-4xy^3+y^4$? *Root* $x^2-2xy+y^2$.

20. Required the square root of $9x^2-3x+\frac{1}{4}$[i]? *Root* $3x-\frac{1}{2}$.

[i] That this part of the subject may not appear imperfect, the following rule for extracting the roots of powers in general is here given, as it may be useful for finding the cube root; but it is too laborious for roots of a higher denomination.

SURDS[k].

58. A surd, or irrational quantity, is a quantity under a radi-

---

RULE I. Find the root of the first term, and place it in the quotient.

II. Subtract its power from that term, and bring down the second term for a dividend.

III. Involve the root last found to the next lower power, and multiply it by the index of the given power, for a divisor.

IV. Divide the dividend by the divisor, and the quotient will be the next term of the root.

V. Involve the whole root, subtract, divide, and proceed as before, until the whole is finished.

EXAMPLES.

1. Required the cube root of $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$?

$x^6+6x^5+15x^4+20x^3+15x^2+6x+1$ (x^2+2x+1 *the root.*

x^6 .... $=$ *cube of* x^2.

$3x^4$) $6x^5$ .... $=$ *second term* $\div 3x^4$.

$x^6+6x^5+12x^4+8x^3$ .... $=$ *cube of* x^2+2x.

$3x^4$) $3x^4$ .... $=$ *remainder* $\div 3x^4$.

$x^6+6x^5+15x^4+20x^3+15x^2+6x+1$ $=$ *cube of* x^2+2x+1.

2. Extract the cube root of $a^3+3a^2b+3ab^2+b^3$. *Root* $a+b$.

3. Required the cube root of $x^3-15x^2+75x-125$? *Root* $x-5$.

4. Required the cube root of $x^6+8x^5-40x^3+96x-64$? *Root* x^2+2x-4.

5. What is the fifth root of $x^5-10x^4+40x^3-80x^2+80x-32$? *Root* $x-2$.

The following method is convenient for some of the higher roots.

For the biquadrate root, extract the square root of the square root.

For the sixth root, ... the square root of the cube root.

For the eighth root, ... the square root, of the square root, of the square root.

For the ninth root, ... the cube root of the cube root, &c. &c.

And, in general, all powers whose indices are any powers or products of 2 and 3, may be unfolded by the rules for the square and cube root; but the 5th, 7th, &c. roots, must be found directly by the above rule, or by some of the methods of approximation. Another method is, by finding the roots of some of the most simple terms, (Art. 56.) and connecting them by the sign $+$ or $-$: then involve this compound root to the required power, which, if equal to the given power, the root is found; but if it differ in some of the signs, let the signs of one or more terms of the root be changed from $+$ to $-$, or from $-$ to $+$, till its power agrees in all respects with the given power.

[k] If the learner should find what is here given on surds too difficult, he may omit either the whole, or any part, for the present; he should however resume the subject as soon as he has passed through quadratic equations.

cal sign or fractional index, the root of which cannot be exactly obtained.

Simple surds are such as are expressed by one single term, as $\sqrt{2}$, $\sqrt[3]{5}$, $\overline{3a}|^{\frac{1}{2}}$, &c.

Compound surds are such as consist of two or more simple surds, connected by the sign + or −; thus $\sqrt{3}+\sqrt{2x}$, $\sqrt[3]{7}-\sqrt[3]{2}$, $\sqrt[n]{\overline{\sqrt{3a}+\sqrt{5}}}$, &c. are compound surds: the latter is called an universal root[l].

REDUCTION OF SURDS[m].

59. *To reduce a rational quantity to the form of a surd.*

RULE. Involve the given quantity to a power equivalent to that of the surd, over which place the index of the surd, and it will be the form required[n].

EXAMPLES.

1. Reduce 3 to the form of the square root.

Thus, $\overline{3}|^2=3\times3=9$. *Wherefore* $\sqrt{9}$ *is the answer required.*

2. Reduce 2 to the form of the cube root, and $3x$ to the form of the fourth root.

---

[l] A surd is a quantity incommensurate to unity, or that is inexpressible in rational numbers by any known method of notation, otherwise than by its proper radical sign or fractional index; hence these numbers are called *irrational* or *incommensurable* numbers, and sometimes *imperfect powers.*

When it is proposed to extract any root from a quantity which is a complete power of the same name with the root, such root can be exactly obtained; but if the given quantity be not a complete power of that name, then the proposed root (which cannot be exactly found) is *denoted* by placing the sign or index over the quantity: this expression (as we have observed) is what is properly called a surd.

The fractional index is mostly to be preferred in practice to the radical sign, because all the rules of fractions may be conveniently applied to fractional indices, whereby the operations are rendered extremely perspicuous and easy.

[m] Reduction of surds does not alter their value, it merely changes them from one form to another; a process which is frequently necessary to prepare them for operations, and for estimating their value.

[n] The reason of this rule is extremely plain, for if any quantity be involved to a power, that quantity is (not only equal to, but is) the root of the power; thus $3=(\sqrt{3^2}=)\sqrt{9}$; $x=\sqrt[4]{x^4}$; $y=\sqrt[n]{y^n}$, &c.

Thus, $\overline{2}|^3=2\times2\times2=8$. *Then* $\overline{8}|^{\frac{1}{3}}$, *the answer.*
And $\overline{3x}|^4=3x\times3x\times3x\times3x=81x^4$, *and* $\overline{81x^4}|^{\frac{1}{4}}$, *the ans.*

3. Reduce $\overline{a}|^{\frac{1}{4}}$ to the form of the square root.
Thus, $\overline{a^{\frac{1}{4}}}|^2=a^{\frac{2}{4}}=a^{\frac{1}{2}}$. *Then* $\sqrt{a^{\frac{1}{2}}}$, *the answer.*

4. Reduce $a-x$ to the form of the square root.
Thus, $\overline{a-x}|^2=a^2-2ax+x^2$; *then* $\sqrt{a^2-2ax+x^2}$, *the ans.*

5. Reduce 7 to the form of the square root, and $\frac{1}{2}a$ to the form of the cube root. *Ans.* $\sqrt{49}$ *and* $\sqrt[3]{\frac{1}{8}a^3}$.

60. *To put the coefficient of a surd under the radical sign.*

RULE I. Reduce the coefficient to the form of the given surd, by the preceding rule.

II. Multiply the quantities under the radical sign by the reduced coefficient, and the product will be the answer *.

6. Given $2\sqrt{x}$, in order to put the coefficient under the radical sign.

First reducing 2 to the form of the square root, we have $2=\sqrt{4}$, *then multiplying x by 4, and placing the common radical sign over the product, we have* $\sqrt{4x}$ *for the answer.*

7. Introduce the coefficients of $3\sqrt[3]{a}$ and $2\sqrt[5]{a-x}$ under the radical sign.

Thus, $3=\overline{3\times3\times3}|^{\frac{1}{3}}=\sqrt[3]{27}$, *then* $\sqrt[3]{27a}$, *the answer.*
And $2=\overline{2\times2\times2\times2\times2}|^{\frac{1}{5}}=\sqrt[5]{32}$; *then* $\sqrt[5]{32.\overline{a-x}}$, *or* $\sqrt[5]{32a-32x}$, *the answer.*

8. Given $5\sqrt{2}$, $3a\sqrt[3]{5x}$, $xy\sqrt[4]{3}$, and $a\sqrt[4]{2+x-z}$, to place the coefficients under the radical sign. *Ans.* $\sqrt{50}$, $\sqrt[3]{135a^3x}$, $\sqrt[4]{3x^4y^4}$, and $\sqrt[4]{a^4\times\overline{2+x-z}}$.

61. *To reduce dissimilar surds to equivalent ones, having a common index.*

RULE I. Reduce the indices to fractions having a common denominator, and place each new numerator over the quantity to which it belongs, for a new index.

---

* This rule is equally obvious with the preceding; for it is evident that $2\times\sqrt{x}=\sqrt{4}\times\sqrt{x}=\sqrt{4x}$, as in the 6th example. Let $x=9$, then $2\sqrt{x}=\sqrt{4}\times\sqrt{9}=\sqrt{4\times9}=\sqrt{36}=6=2\times3$; whence $2\sqrt{x}=\sqrt{4x}$.

II. Write 1 over the common denominator, and place this fraction as an index over the given quantities, with their new indices; if those quantities are numbers, involve them to the power denoted by the new index, over which place the said fraction for an index [p].

9. Reduce $\overline{3}|^{\frac{2}{3}}$ and $\overline{2}|^{\frac{3}{4}}$ to equivalent surds, having a common index.

OPERATION.

$$\left.\begin{array}{r} \textit{First } 2\times4=8 \\ 3\times3=9 \end{array}\right\} \textit{ new numerators.}$$

And $3\times4=12$ *common denominator.*

$$\textit{Then } \overline{3^8}|^{\frac{1}{12}}=\overline{3\times3\times3\times3\times3\times3\times3\times3}|^{\frac{1}{12}}=\overline{6561}|^{\frac{1}{12}} \textit{ ans.}$$

$$\textit{Also } \overline{2^9}|^{\frac{1}{12}}=\overline{2\times2\times2\times2\times2\times2\times2\times2\times2}|^{\frac{1}{12}}=\overline{512}|^{\frac{1}{12}}, \textit{ ans.}$$

Explanation.

I first reduce $\frac{2}{3}$ and $\frac{3}{4}$ to a common denominator, I place the numerator 8 over the quantity 3, and the numerator 9 over 2; then putting 1 for a numerator over 12, I place this fraction over 3^8 and 2^9, making $\overline{3^8}|^{\frac{1}{12}}$ and $\overline{2^9}|^{\frac{1}{12}}$; I then involve 3 to the 8th power, and 2 to the 9th; over each of which I put the index $\frac{1}{12}$ for the answers.

10. Reduce $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ to equal surds, having a common index.

First $1\times3=3$ *new index of* x.

$1\times2=2$ .... *of* y.

$2\times3=6$ *common denominator.*

Wherefore $\overline{x^3}|^{\frac{1}{6}}$ *and* $\overline{y^2}|^{\frac{1}{6}}$ *are the surds required.*

11. Reduce $\overline{x+y}|^{\frac{1}{2}}$ and $\overline{x-y}|^{\frac{1}{3}}$ to a common index.

Thus $1\times3=3$ *index of* $x+y$.

$1\times2=2$ .... *of* $x-y$.

$2\times3=6$ *common denominator.*

Therefore $\overline{(x+y)^3}|^{\frac{1}{6}}=\overline{x^3+3x^2y+3xy^2+y^3}|^{\frac{1}{6}}$, *ans.*

And $\overline{(x-y)^2}|^{\frac{1}{6}}=\overline{x^2-2xy+y^2}|^{\frac{1}{6}}$, *ans.*

---

[p] The reason of this rule will appear from the 9th example; for $\frac{2}{3}$ and $\frac{3}{4}$ reduced to a common denominator, become respectively $\frac{8}{12}$ and $\frac{9}{12}$; wherefore $3^{\frac{2}{3}}$ becomes $3^{\frac{8}{12}}$ and $2^{\frac{3}{4}}$ becomes $2^{\frac{9}{12}}$: but $3^{\frac{8}{12}}$ implies the 12th root of 3 in the 8th power, or $\overline{3^8}|^{\frac{1}{12}}=\overline{6561}|^{\frac{1}{12}}$; and $2^{\frac{9}{12}}$ implies in like manner $\overline{2^9}|^{\frac{1}{12}}=\overline{512}|^{\frac{1}{12}}$, as in the example, and the like of other quantities; wherefore the rule is manifest.

12. Reduce $7^{\frac{3}{4}}$ and $8^{\frac{1}{2}}$ to a common index. *Ans.* $\overline{117649}|^{\frac{1}{8}}$ *and* $\overline{4096}^{\frac{1}{8}}$.

13. Reduce $x^{\frac{1}{4}}$, $y^{\frac{1}{3}}$ and $z^{\frac{1}{2}}$ to a common index. *Ans.* $\overline{x^3}|^{\frac{1}{12}}$, $\overline{y^4}|^{\frac{1}{12}}$, *and* $\overline{z^6}|^{\frac{1}{12}}$.

14. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to a common index. *Ans.* $\overline{a^m}|^{\frac{1}{mn}}$ *and* $\overline{b^n}|^{\frac{1}{mn}}$.

15. Reduce $\overline{a+x}|^{\frac{2}{3}}$ and $z^{\frac{1}{2}}$ to a common index.
Ans. $\overline{a^4+4a^3x+6a^2x^2+4ax^3+x^4}|^{\frac{1}{6}}$ *and* $\overline{z^3}|^{\frac{1}{6}}$.

62. *To reduce surds to equivalent ones having a given index.*

RULE I. Divide the indices of the surds by the given index, and place the quotients each over its proper quantity, for a new index.

II. Over the given quantities with their new indices, write the given index, the results will be the surds required[q].

16. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to equal surds, having the common index $\frac{1}{4}$.

OPERATION.

$$\textit{First } \frac{1}{2}\div\frac{1}{4}=\frac{1}{2}\times\frac{4}{1}=\frac{4}{2}=2 \textit{ the index of } 2.$$

$$\textit{Then } \frac{1}{3}\div\frac{1}{4}=\frac{1}{3}\times\frac{4}{1}=\frac{4}{3} \textit{ the index of } 3.$$

Wherefore $\overline{2^2}|^{\frac{1}{4}}=\overline{4}|^{\frac{1}{4}}$ *and* $\overline{3^{\frac{4}{3}}}|^{\frac{1}{4}}=\overline{81^{\frac{1}{3}}}|^{\frac{1}{4}}$, *are the quantities required.*

Explanation.

I first divide $\frac{1}{2}$ and $\frac{1}{3}$ each by $\frac{1}{4}$, and the quotients are 2 and $\frac{4}{3}$; these I place as new indices respectively over 2 and 3, and over these the given index $\frac{1}{4}$. I then involve 2 to the square, and 3 to the 4th power, by which the new index of the former is taken away, and that of the latter reduced to $\frac{1}{3}$; over the results I place the given index $\frac{1}{4}$ for the answer.

17. Reduce $a^{\frac{1}{m}}$ and $z^{\frac{1}{n}}$ to equal surds, having $\frac{r}{s}$ for a common index.

$$\frac{1}{m}\div\frac{r}{s}=\frac{1}{m}\times\frac{s}{r}=\frac{s}{mr}=\textit{index of } a.$$

---

[q] Dividing the indices is equivalent to extracting the root denoted by the divisor; and placing the divisor as an index over the result is equivalent to involving the result to the power denoted by the divisor: wherefore since equal evolution and involution take place, the value of the given quantity is not altered by the transformation effected under this rule; this is evident from Ex. 16. where $2^{\frac{1}{2}}$ is evidently equal to $\sqrt[4]{\overline{2^{\frac{1}{2}}}|^4}=\sqrt[4]{2^{\frac{4}{2}}}=\sqrt[4]{2^2}=\overline{4}|^{\frac{1}{4}}$, and $3^{\frac{1}{3}}$ is in like manner evidently equal to $\sqrt[4]{\overline{3^{\frac{1}{3}}}|^4}=\sqrt[4]{3^{\frac{4}{3}}}=\sqrt[4]{\overline{81}|^{\frac{1}{3}}}=\overline{81^{\frac{1}{3}}}|^{\frac{1}{4}}$, all which is sufficiently plain from Art. 60.

$$\frac{1}{n} \div \frac{r}{s} = \frac{1}{n} \times \frac{s}{r} = \frac{s}{nr} = \textit{index of } z.$$

Wherefore $(a^{\frac{s}{nr}})^{\frac{r}{s}}$ and $(z^{\frac{s}{nr}})^{\frac{r}{s}}$, *the answer.*

18. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{6}}$ to equal surds, having the common index $\frac{1}{3}$. *Ans.* $\overline{2}|^{\frac{1}{3}}$ *and* $\overline{3^{\frac{1}{2}}}|^{\frac{1}{3}}$.

19. Reduce $x+y$ and $3^{\frac{1}{4}}$ to equal surds, having $\frac{1}{2}$ for a common index. *Ans.* $\overline{x^2+2xy+y^2}|^{\frac{1}{2}}$ *and* $\overline{3^{\frac{1}{2}}}|^{\frac{1}{2}}$.

20. Reduce $x^{\frac{1}{n}}$, $y^{\frac{1}{m}}$ *and* $z^{\frac{1}{r}}$ to equivalent quantities, having s for a common index. *Ans.* $\overline{x^{\frac{1}{ns}}}|^{s}$, $\overline{y^{\frac{1}{ms}}}|^{s}$, *and* $\overline{z^{\frac{1}{rs}}}|^{s}$.

63. *To reduce surds to their simplest terms.*

RULE I. Divide the given surd by the greatest power (of the same name with it) that will divide it without remainder, and place the said power and the quotient, with the sign $\times$ between them, under the radical sign.

II. Extract the root of the fore-mentioned power, and place its root before the said quotient, with the proper radical sign between them [r].

21. Reduce $\sqrt{32}$ to its simplest terms.

OPERATION.

$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$, *the answer.*

Explanation.

The greatest square that will divide 32 is 16, and the quotient is 2; the root of 16 is 4, therefore $4\sqrt{2}$ is the answer.

---

[r] In this rule the given surd is resolved into two factors, one of which is a power of the same name with the surd.

Now it is evidently the same thing to multiply the remaining factor by this power, both being under the radical sign, or to multiply the factor *under the sign* by the root of the power *not under the sign;* thus, Ex. 21. $\sqrt{32}$ is evidently equal to $\sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2}$, but $\sqrt{16} = 4$, wherefore $\sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$, as in the example; wherefore, the transformation which takes place under this rule does not alter the value of the given surd, which was to be shewn.

22. Reduce $\sqrt[3]{108x^4y^5}$ to its simplest terms.

$\sqrt[3]{108x^4y^5} = \sqrt[3]{27a^3y^3 \times 4xy^2} = 3ay^3\sqrt{4xy^2}$. *Ans.*

23. Reduce $\sqrt{8x^3-12x^2y}$ to its simplest terms.

$\sqrt{8x^3-12x^2y} = \sqrt{4x^2 \times 2x-3y} = 2x\sqrt{2x-3y}$. *Ans.*

24. Reduce $\sqrt{50}$ to its simplest terms. *Ans.* $5\sqrt{2}$.

25. Reduce $\sqrt{24}$ to its simplest terms. *Ans.* $2\sqrt{6}$.

26. Reduce $\sqrt[3]{4a^3xy^4}$ and $3\sqrt[4]{16x^5}$ to their simplest terms. *Ans.* $ay^3\sqrt{4xy}$ *and* $12x^4\sqrt{x}$.

27. Reduce $\overline{3x^5-x^4y+32x^4z^2}|^{\frac{1}{4}}$ and $\frac{3}{5}\sqrt[3]{1000x^7}$ to their simplest terms. *Ans.* $x^4\sqrt{3.x-y+32z^2}$ *and* $6x^2\sqrt[3]{x}$.

64. *To reduce a fractional surd to its equivalent integral one.*

RULE I. Multiply the numerator under the radical sign by that power of its denominator, whose index is 1 less than the index of the surd.

II. Take the denominator away from under the radical sign, and divide the coefficient by it, and the surd part will be an integral quantity, which must be reduced to its simplest terms by the foregoing rule *.

---

* To prove the truth of this rule, let $a\sqrt[3]{\frac{b}{c}}$ be a given surd, of which it is required to reduce the radical part $\sqrt[3]{\frac{b}{c}}$ to an integer, without altering the value of the given quantity. Now it is evident, that if both terms of a fraction be multiplied by the same quantity, (let that quantity be whatever it may,) the value of the fraction is not altered; wherefore let both terms of the given surd be multiplied by c^2, and it will become $a\sqrt[3]{\frac{bc^2}{c^3}}$; now the denominator c^3 is a complete power of the same denomination with the surd, and therefore (Art. 63.) it may be taken away, provided its root c be made the divisor of the coefficient a; (for dividing by c is the same as dividing by $\sqrt[3]{c^3}$; and dividing either one factor, the other factor, or the product, by the same or equal quantities, produces in each case the same result;) wherefore the given surd $a\sqrt[3]{\frac{b}{c}}$ becomes successively $a\sqrt[3]{\frac{bc^2}{c^3}}$ and $\frac{a}{c}\sqrt[3]{bc^2}$; in the latter of which the surd part is reduced to an integer, and the whole $\frac{a}{c}\sqrt[3]{bc^2}$ is of the same value with the given surd $a\sqrt[3]{\frac{b}{c}}$, which was to be shewn.

28. **Reduce the radical part of $\frac{2}{3}\sqrt{\frac{1}{3}}$ to a whole number.**

First, the numerator 1 *being multiplied by its denominator* 3, *produces* 3 *for the surd; then, dividing the coefficient* $\frac{2}{3}$ *by* 3, *the quotient is* $\frac{2}{9}$; *wherefore* $\frac{2}{9}\sqrt{3}$ *is the answer.*

29. **Reduce the radical part of $\frac{3}{4}\sqrt{\frac{4}{5}}$ to a whole number.**

Multiplying the numerator 4 *by* 5, *gives* 20 *for the surd; then dividing the coefficient* $\frac{3}{4}$ *also by* 5, *gives* $\frac{3}{20}$ *for the coefficient; whence* $\frac{3}{20}\sqrt{20}$ *is the surd, which reduced to its simplest terms gives* $\frac{3}{20}\sqrt{20}=\frac{3}{20}\sqrt{4\times5}=\frac{6}{20}\sqrt{5}=\frac{3}{10}\sqrt{5}$ *for the answer.*

30. **Reduce $\sqrt[3]{\frac{2a}{5x}}$ to an integral surd.**

Multiply $2a$ *by (the square of* $5x$ *or)* $25x^2$, *and the product* $50ax^2$ *will be the surd part; then divide the coefficient* 1 *(understood) by* $5x$, *and the quotient* $\frac{1}{5x}$ *is the coefficient: wherefore* $\frac{1}{5x}\sqrt[3]{50ax^2}$ *is the answer.*

31. **Reduce $\sqrt{\frac{50}{147}}$ and $\sqrt[3]{\frac{3}{4}}$ to equivalent surds, having the radical parts whole numbers.** *Ans.* $\frac{5}{21}\sqrt{6}$ *and* $\frac{1}{2}\sqrt[3]{6}$.

32. **Reduce the radical parts of $4\sqrt{\frac{3}{7}}$, $\sqrt[3]{\frac{12x}{13a^4}}$, and $3\sqrt[4]{\frac{2z^2}{7y^3}}$ to integers.** *Ans.* $\frac{4}{7}\sqrt{21}$, $\frac{1}{13a^2}\sqrt[3]{2028a^2x}$, *and* $\frac{3}{7y}\sqrt[4]{686z^2y}$.

33. **Reduce the radical parts of $\sqrt[3]{\frac{32z}{250x^3}}$ and $\sqrt{\frac{150}{441}}$ to whole numbers.** *Ans.* $\frac{2}{5x}\sqrt[3]{2xz}$ *and* $\frac{5}{21}\sqrt{6}$.

65. ADDITION OF SURDS.

RULE I. Reduce the quantities to a common index, the

fractions to a common denominator, (Art. 180. Part I.) and the surds to their simplest terms, Art. 63.

II. If the surd part be alike in all the quantities, add the coefficients together, and to their sum subjoin the common surd.

III. If the surd parts be unlike, the quantities cannot be added, otherwise than by connecting them together by means of their signs [t].

EXAMPLES.

1. Add $\sqrt{8}$ and $\sqrt{18}$ together.

These reduced to their simplest terms, are,

$$\sqrt{8} = \sqrt{4\times 2} = 2\sqrt{2}$$
$$\sqrt{18} = \sqrt{9\times 2} = 3\sqrt{2}$$

And by adding . . . $\overline{5\sqrt{2}}$ *the sum required.*

2. Add $\sqrt[3]{24\,a}$, $\sqrt[3]{192\,a}$, and $\sqrt[3]{81\,a}$ together.

Thus $\sqrt[3]{24\,a} = \sqrt[3]{8\times 3\,a} = 2\sqrt[3]{3\,a}.$

$\sqrt[3]{192\,a} = \sqrt[3]{64\times 3\,a} = 4\sqrt[3]{3\,a}.$

$\sqrt[3]{81\,a} = \sqrt[3]{27\times 3\,a} = 3\sqrt[3]{3\,a}.$

Their sum $\overline{9\sqrt[3]{3\,a}}$ *the answer.*

3. Add $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{6}}$, and $\sqrt{\frac{8}{27}}$ together.

These reduced first to a common denominator, and then to their simplest terms, become

$$\sqrt{\frac{1}{3}} = \sqrt{\frac{54}{162}} = \sqrt{\frac{9\times 6}{81\times 2}} = \frac{3}{9}\sqrt{3} = \frac{1}{3}\sqrt{3}.$$
$$\sqrt{\frac{1}{6}} = \sqrt{\frac{27}{162}} = \sqrt{\frac{9\times 3}{81\times 2}} = \frac{3}{9}\sqrt{\frac{3}{2}}.$$
$$\sqrt{\frac{8}{27}} = \sqrt{\frac{48}{162}} = \sqrt{\frac{16\times 3}{81\times 2}} = \frac{4}{9}\sqrt{\frac{3}{2}}.$$

Their sum $\overline{\frac{7}{9}\sqrt{\frac{3}{2}} + \frac{1}{3}\sqrt{3}} = \frac{7}{18}\sqrt{6} + \frac{1}{3}\sqrt{3}.$

---

[t] This rule is sufficiently plain, without any further illustration than what is contained in the notes on addition of rational quantities, Art. 36. to 40.

4. Add $\sqrt{8}$ and $\sqrt{32}$ together. *Sum* $6\sqrt{2}$.

5. Add $\sqrt{4x^2z}$ and $\sqrt{16x^2z}$ together. *Sum* $6x\sqrt{z}$.

6. Add $\sqrt[3]{32}$, $\sqrt[3]{500}$, and $2\sqrt[3]{4}$ together. *Sum* $9\sqrt[3]{4}$.

7. Add $\sqrt[3]{8x^3y^2}$ and $\sqrt[3]{16x^3y^2}$ together. *Sum* $2x\sqrt[3]{y^2}+2x\sqrt[3]{2y^2}$.

8. Add $3\sqrt[5]{64a^5z-32a^6z^2}$ and $2\sqrt[5]{64a^5z-32a^6z^2}$ together. *Sum* $10a\sqrt[5]{2z-az^2}$.

9. Add $2\sqrt[4]{x^6}$, $3\sqrt[4]{x^3}$, and $4\sqrt[4]{x^5}$ together. *Sum* $2x\sqrt[4]{x^2}+3\sqrt[4]{x^3}+4x\sqrt[4]{x}$.

10. Find the sum of $\sqrt{\frac{20}{27}}$ and $\sqrt{\frac{5}{12}}$. *Sum* $\frac{7}{18}\sqrt{15}$.

66. SUBTRACTION OF SURDS.

RULE I. Prepare the quantities (if they require it) as in addition.

II. Consider which surd is to be subtracted, and, if both surd parts are alike, subtract its coefficient from the coefficient of the other, subjoining the common surd to the remainder.

III. But if after the necessary reduction the surd parts are unlike, change the sign of the quantity to be subtracted, and then connect it with the other quantity [*].

EXAMPLES.

1. From $3\sqrt{28}$ take $\sqrt{63}$.

These reduced to their simplest terms, are,

$3\sqrt{28}=3\sqrt{4\times7}=3\times2\sqrt{7}=6\sqrt{7}$.

and $\sqrt{63}=\sqrt{9\times7}=3\sqrt{7}$

their difference$=3\sqrt{7}$ *the Answer.*

2. From $4\sqrt[3]{128a^4}$ take $2\sqrt[3]{16a^4}$

Thus $4\sqrt[3]{128a^4}=4\sqrt[3]{64a^3\times2a}=16a\sqrt[3]{2a}$

$2\sqrt[3]{16a^4}=2\sqrt[3]{8a^3\times2a}=\ \ 4a\sqrt[3]{2a}$

Diff. $12a\sqrt[3]{2a}$ *Ans.*

3. From $\sqrt{\frac{2}{3}}$ subtract $\sqrt{\frac{27}{50}}$.

---

[*] Subtraction of surds evidently depends on the same principles with subtraction of rational quantities, as will readily appear from a bare contemplation of the rule.

First reducing these to a common denominator,

$\sqrt{\frac{2}{3}}=\sqrt{\frac{100}{150}}$, *and* $\sqrt{\frac{27}{50}}=\sqrt{\frac{81}{150}}$.

Then to their simplest terms,

$$\sqrt{\frac{100}{150}}=\sqrt{\frac{100\times 1}{25\times 6}}=\frac{10}{5}\sqrt{\frac{1}{6}}, \text{ and } \sqrt{\frac{81}{150}}=\sqrt{\frac{81\times 1}{25\times 6}}=\frac{9}{5}\sqrt{\frac{1}{6}}.$$

Then by subtracting $\frac{10}{5}\sqrt{\frac{1}{6}}-\frac{9}{5}\sqrt{\frac{1}{6}}=\frac{1}{5}\sqrt{\frac{1}{6}}$; *which by making the surd part a whole number, becomes* $\frac{1}{5\times 6}\sqrt{1\times 6}=\frac{1}{30}\sqrt{6}$, *the answer.*

4. From $\sqrt{50}$ take $\sqrt{18}$. *Diff.* $2\sqrt{2}$.
5. From $3\sqrt{175}$ take $2\sqrt{28}$. *Diff.* $11\sqrt{7}$.
6. From $\sqrt[3]{250a^3x}$ take $\sqrt[3]{16a^3x}$. *Diff.* $3a\sqrt[3]{2x}$.
7. From $\sqrt{8a}$ take $\sqrt{8a^2}$. *Diff.* $2\sqrt{2a}-2a\sqrt{2}$.
8. From $\sqrt[3]{\frac{2}{3}}$ take $\sqrt[3]{\frac{9}{32}}$. *Diff.* $\frac{1}{12}\sqrt[3]{18}$.
9. From $3\sqrt[3]{189x^3y+27x^3}$ take $4\sqrt[3]{56x^3y+8x^3}$. *Diff.* $x\sqrt[3]{7y+1}$.

67. MULTIPLICATION OF SURDS.

RULE I. Reduce the surds to the same index, (if they require it,) by Art. 61. then multiply the coefficients together for the rational part of the product.

II. Multiply the surd parts together, and having placed the radical sign over the product, subjoin it to the former product, and reduce the surd to its simplest terms [x]. Art. 63.

EXAMPLES.

1. Multiply $4\sqrt{2}$ by $5\sqrt{8}$.

Thus $4\times 5=20$, *the coefficient, and* $\sqrt{2}\times\sqrt{8}=\sqrt{16}$, *the*

---

[x] To shew that this mode of operation agrees with known principles, let example 1 be proposed, where $4\sqrt{2}$ is to be multiplied by $5\sqrt{8}$; let the coefficients be put under the radical sign, (Art. 60.) and these quantities become $\sqrt{32}$ and $\sqrt{200}$; wherefore $\sqrt{32}\times\sqrt{200}=\sqrt{6400}=80$, as in the example. Again, ex. 2, where $5\sqrt{6}$ is to be multiplied by $4\sqrt{3}$, proceeding as before, we have $\sqrt{150}\times\sqrt{48}=\sqrt{7200}=\sqrt{3600\times 2}$ (Art. 63.) $=60\sqrt{2}$, as in the example, and the like may be shewn in all other cases.

surd part: therefore $20\sqrt{16}$, *or* $20\sqrt{4\times 4}=20\times 4=80$, *the answer.*

2. Multiply $5\sqrt{6}$ by $4\sqrt{3}$.

Thus $5\sqrt{6}\times 4\sqrt{3}=20\sqrt{18}=20\sqrt{9\times 2}=20\times 3\sqrt{2}=60\sqrt{2}$, *the product.*

3. Multiply $\frac{2}{3}\sqrt{\frac{5}{7}}$ and $\frac{3}{4}\sqrt{\frac{1}{2}}$ together.

Thus $\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$ *the rational part.*

And $\sqrt{\frac{5}{7}}\times\sqrt{\frac{1}{2}}=\sqrt{\frac{5}{14}}$ *the surd part.*

Then $\sqrt{\frac{5}{14}}$ *reduced to an integral surd is* $\frac{1}{14}\sqrt{70}$.

Wherefore $\frac{1}{2}\times\frac{1}{14}\sqrt{70}$ *is* $\frac{1}{28}\sqrt{70}$, *the product required.*

4. Multiply $2+\sqrt{3}\,a$ and $3-\sqrt{2}\,y$ together.

$$\begin{array}{l} \textit{Thus } 2+\sqrt{3}\,a \\ \quad 3-\sqrt{2}\,y \\ \hline 6+3\sqrt{3a}-2\sqrt{2}\,y-\sqrt{6}\,ay, \textit{ the product.} \\ \hline \end{array}$$

5. Multiply $\overline{x+y}|^{\frac{1}{2}}$ and $\overline{x+y}|^{\frac{1}{3}}$ together.

These reduced to a common index, become

$\overline{x+y}|^{\frac{1}{2}}=\overline{x+y}|^{\frac{3}{6}}=\overline{(x+y)^3}|^{\frac{1}{6}}$, *and* $\overline{x+y}|^{\frac{1}{3}}=\overline{x+y}|^{\frac{2}{6}}=\overline{(x+y)^2}|^{\frac{1}{6}}$.

Wherefore $\overline{(x+y)^3}|^{\frac{1}{6}}\times\overline{(x+y)^2}|^{\frac{1}{6}}=\overline{(x+y)^5}|^{\frac{1}{6}}=$

$\overline{x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+x^5}|^{\frac{1}{6}}$, *the product required.*

6. Multiply $2\sqrt{8}$ by $3\sqrt{6}$. *Prod.* $24\sqrt{3}$.
7. Multiply $\sqrt{2}$, $2\sqrt{3}$, and $3\sqrt{5}$ together. *Prod.* $6\sqrt{30}$.
8. Multiply $2^3\sqrt{a^2}$ and $4^3\sqrt{a^2y}$ together. *Prod.* $8a^3\sqrt{ay}$.
9. Multiply $x^{\frac{3}{4}}$ and $x^{\frac{1}{2}}$ together. *Prod.* $x^4\sqrt{x}$.
10. Multiply $\frac{2}{3}\sqrt{\frac{1}{2}}$ and $\frac{1}{2}\sqrt{\frac{5}{6}}$ together. *Prod.* $\frac{1}{18}\sqrt{15}$.
11. Multiply $a+\sqrt{z}$ and $a-\sqrt{z}$ together. *Prod.* a^2-z.

68. DIVISION OF SURDS.

RULE. Reduce the surds to the same index, (Art. 61.) if they require it, then divide the rational parts by the rational, and the surd by the surd; the former quotient annexed to the latter

will be the quotient, which must be reduced to its simplest terms as before [y].

EXAMPLES.

1. Divide $20\sqrt{21}$ by $4\sqrt{3}$.

Thus $\frac{20}{4}=5$ *the rational part of the quotient.*

And $\frac{\sqrt{21}}{\sqrt{3}}=\sqrt{7}$ *the surd part.*

Wherefore $5\sqrt{7}$ *is the quotient required.*

2. Divide $12^3\sqrt{48}$ by $6^3\sqrt{2}$.

Thus $\frac{12^3\sqrt{48}}{6^3\sqrt{2}}=2^3\sqrt{24}=2^3\sqrt{8\times 3}=4^3\sqrt{3}$, *the quotient.*

3. Divide $\frac{2}{3}\sqrt[4]{\frac{3}{4}}$ by $\frac{1}{2}\sqrt[4]{\frac{2}{3}}$.

Thus $\frac{2}{3}\div\frac{1}{2}=\frac{2}{3}\times\frac{2}{1}=\frac{4}{3}$ *the rational part.*

And $\sqrt[4]{\frac{3}{4}}\div\sqrt[4]{\frac{2}{3}}=\sqrt[4]{\frac{3}{4}}\times\sqrt[4]{\frac{3}{2}}=\sqrt[4]{\frac{9}{8}}$ *the surd part.*

Whence $\frac{4}{3}\sqrt[4]{\frac{9}{8}}=\frac{4}{3\times 8}\sqrt[4]{9\times 8^3}=\frac{4}{24}\sqrt[4]{4608}=\frac{1}{6}$ $\sqrt[4]{256\times 18}=\frac{4}{6}\sqrt[4]{18}=\frac{2}{3}\sqrt[4]{18}$, *the quotient required.*

4. Divide $x^3-x^2z\sqrt{z}$ by $x-x\sqrt{z}$.

$$
\begin{array}{l}
x-x\sqrt{z})\,x^3-x^2z\sqrt{z}\,(x^2+x^2\sqrt{z}+x^2z. \quad \textit{Quotient.} \\
\qquad\qquad\underline{x^3-x^3\sqrt{z}} \\
\qquad\qquad\qquad x^3\sqrt{z} \\
\qquad\qquad\qquad \underline{x^3\sqrt{z}-x^3z} \\
\qquad\qquad\qquad\qquad x^3z-x^2z\sqrt{z} \\
\qquad\qquad\qquad\qquad \underline{x^3z-x^3z\sqrt{z}}
\end{array}
$$

[y] Division being the converse of multiplication, its method of operation, which is manifestly the converse of that of multiplication, must needs be true; but it may likewise be shewn to be so in a similar manner to that employed in the preceding note: thus ex. 1. where $20\sqrt{21}$ is required to be divided by $4\sqrt{3}$; putting the coefficients of both under the radical sign, (Art. 60.) and dividing the former by the latter, we have $\frac{\sqrt{8400}}{\sqrt{48}}=\sqrt{175}=\sqrt{25\times 7}$ (Art. 63.) $=5\sqrt{7}$, as in the example referred to; and the same may be shewn to hold true in all other cases.

5. Divide $12\sqrt{15}$ by $4\sqrt{3}$. *Quot.* $3\sqrt{5}$.
6. Divide $30\sqrt{10}$ by $5\sqrt{5}$. *Quot.* $6\sqrt{2}$.
7. Divide $\sqrt[3]{35a^2b^2}$ by $\sqrt[3]{7a}$. *Quot.* $\sqrt[3]{5ab^2}$.
8. Divide $3\sqrt{8}$ by $2\sqrt{2}$. *Quot.* 3.
9. Divide $\frac{1}{3}\sqrt{\frac{5}{12}}$ by $\frac{2}{3}\sqrt{\frac{1}{2}}$. *Quot.* $\frac{1}{12}\sqrt{30}$.
10. Divide $x^{\frac{2}{3}}$ by $x^{\frac{3}{4}}$. *Quot.* $\frac{1}{x^{\frac{1}{12}}}$.
11. Divide $16-\sqrt[3]{9}$ by $4+\sqrt[3]{3}$. *Quot.* $4-\sqrt[3]{3}$.

69. INVOLUTION OF SURDS.

RULE I. Involve the coefficient to the power required, for the rational part of the power, (Art. 265 to 267. Part I.)

II. Multiply the index of the surd by the index of the power to which it is to be raised, and the product will be the surd part.

III. Annex the rational part of the power to the surd part, and the result will be the power required [*].

EXAMPLES.

1. Involve $2\sqrt[3]{x}$ to the fourth power.

Thus $(2)^4=2\times2\times2\times2=16$ *the rational part.*

And $(x^{\frac{1}{3}})^4=x^{\frac{1}{3}\times4}=x^{\frac{4}{3}}=\sqrt[3]{x^4}=\sqrt[3]{x^3\times x}=x\sqrt[3]{x}$ *the surd part.*

Whence $16\times x\sqrt[3]{x}=16x\sqrt[3]{x}$, *the power required.*

2. Involve $\frac{2}{3}\sqrt{5}$ to the third power.

Thus $\left(\frac{2}{3}\right)^3=\frac{8}{27}$ *the rational part.*

And $(\sqrt{5})^3=5^{\frac{1}{2}\times3}=5^{\frac{3}{2}}=(5^3)^{\frac{1}{2}}=(125)^{\frac{1}{2}}=(25\times5)^{\frac{1}{2}}=5\sqrt{5}$ *for the surd part.*

Whence $\frac{8}{27}\times5\sqrt{5}=\frac{40}{27}\sqrt{5}=1\frac{13}{27}\sqrt{5}$, *the answer.*

---

[*] This rule is founded on the same principles with involution of rational quantities, (Art. 52.) and multiplication of surds, Art. 67.

3. Involve $3x - 4y\sqrt{z}$ to the square.

$$\begin{array}{l} \textit{Thus } 3x - 4y\sqrt{z} \\ \underline{\phantom{\textit{Thus }} 3x - 4y\sqrt{z}} \\ 9x^2 - 12xy\sqrt{z} \\ \underline{ - 12xy\sqrt{z} + 16y^2z} \\ 9x^2 - 24xy\sqrt{z} + 16y^2z \textit{ the square required.} \end{array}$$

4. Involve $2\sqrt[3]{2}$ and $3\sqrt{3x}$ each to the square. *Ans.* $4\sqrt[3]{4}$ and $27x$.

5. Involve $\frac{1}{3}\sqrt{\frac{1}{2}}$ and $\frac{3}{4}\sqrt[3]{\frac{a}{3}}$ each to the cube. *Answer* $\frac{1}{108}\sqrt{2}$ *and* $\frac{9a}{64}$.

6. Involve $\frac{2}{3}\sqrt[5]{x}$ and $4\sqrt[6]{a}$ to the square. *Answer* $\frac{4}{9}\sqrt[5]{x^2}$ *and* $16\sqrt[3]{a}$.

7. Involve $1+\sqrt{3}$ to the cube, and $\frac{1}{3}\sqrt[4]{a^3}$ to the fourth power. *Ans.* $10+6\sqrt{3}$ *and* $\frac{1}{81}a^3$.

70. EVOLUTION OF SURDS.

RULE I. Extract the required root of the rational part for the coefficient.

II. Multiply the index of the surd into the (fractional) index of the root to be extracted, for the index of the surd part.

III. Annex the rational part to the surd for the root required [*].

EXAMPLES.

1. Extract the square root of $16\sqrt[3]{9}$.

Thus $\sqrt{16}=4$ *the rational part, and* $9^{\frac{1}{3}}\times\frac{1}{2}=9^{\frac{1}{6}}$ *the surd part; whence* $4\sqrt[6]{9}$ *is the answer.*

2. Required the fourth root of $81a^3y^6z$.

Thus $\sqrt[4]{81}=3$, *and* $\sqrt[4]{a^3y^6z}=a^{\frac{3}{4}}y^{\frac{6}{4}}z^{\frac{1}{4}}=a^{\frac{3}{4}}y^{\frac{3}{2}}z^{\frac{1}{4}}$.

Wherefore $3a^{\frac{3}{4}}y^{\frac{3}{2}}z^{\frac{1}{4}}$ *is the answer.*

---

[*] This rule depends on the same foundation with evolution of rational quantities, Art. 56.

3. Required the square root of $1-2\sqrt{2}+2$?

Thus $1-2\sqrt{2}+2$ ($1-\sqrt{2}$ *the root.*

$$\begin{array}{l} \quad 1 \\ 2-\sqrt{2})\overline{-2\sqrt{2}+2} \\ \qquad\quad \underline{-2\sqrt{2}+2} \end{array}$$

4. Extract the square root of $9^3\sqrt{3}$ and of $\frac{1}{2}a^2$. *Roots* $3^6\sqrt{3}$ *and* $a\sqrt{\frac{1}{2}}$.

5. Required the cube root of $a^4\sqrt{3}$ and of $\frac{27}{64}{}^3\sqrt{\frac{3}{4}}$. *Roots* $a^3\sqrt{a}\times 3^{\frac{1}{6}}$ and $\frac{3}{4}{}^9\sqrt{\frac{3}{4}}$.

6. Required the square root of $x^2-2x\sqrt{y}+y$? *Root* $x-\sqrt{y}$.

EQUATIONS.

71. An algebraic equation [b] is an expression whereby two quantities (either simple or compound) are declared *equal* to each other, by means of the sign $=$ of equality placed between them.

72. In equations consisting of *known* and *unknown* quantities, when the unknown quantity is in the *first power only*, the expression is called a *simple* equation; when it is in the *second power only*, it is named a *pure quadratic*; when in the *third power only*, a *pure cubic*, &c. But when the unknown quantity consists of two or more different powers in the same equation, it is then named an *adfected equation.*

73. REDUCTION OF SIMPLE EQUATIONS, INVOLVING ONE UNKNOWN QUANTITY ONLY.

The business of equations is to find the value of the unknown quantities concerned in the equation, by means of those that are known; this process is called *Reduction*, and its operations are founded on the following self-evident principles: namely, if equals be added to, subtracted from, multiplied into, or divided by equals, the results will respectively be equal.

---

[b] The word *Equation* is derived from the Latin *æquus*, equal.

The reduction of an equation consists in managing its terms so that the unknown quantity may, at the end of the process, stand alone and in its first power, on one side of the sign $=$, and known quantities only on the other: when this is effected, the business is done; for the value of the unknown quantity is found, it being equal to the aggregate of the known quantities incorporated together, according to the import of their signs.

74. *To transpose the terms of an equation, that is, to remove them from one side of the sign $=$ of equality to the other.*

RULE. Make a new equation, in which place the quantity to be transposed on the opposite side of the sign $=$, to that on which it stood in the preceding equation, observing to change its sign from $+$ to $-$, or from $-$ to $+$; and let the rest of the quantities stand as in the preceding equation [c].

EXAMPLES.

1. Given $4+5-2=6+3+7-9$, to transpose the terms.

OPERATION.

1st *step*.	To transpose -2. *thus*	$4+5=6+3+7-9+2$.
2nd *step*.	To transpose $+5$. . . .	$4=6+3+7-9+2-5$.
3rd *step*.	To transpose $+6$. . . .	$4-6=3+7-9+2-5$.
4th *step*.	To transpose $+3$. . . .	$4-6-3=7-9+2-5$.
5th *step*.	To transpose $+7$. . . .	$4-6-3-7=-9+2-5$.
6th *step*.	To transpose -9. . . .	$4-6-3-7+9=2-5$.
7th *step*.	To transpose $+2$. . . .	$4-6-3-7+9-2=-5$.
8th *step*.	To transpose -5. . . .	$4-6-3-7+9-2+5=0$.

---

[c] This rule is founded on the following self-evident principle; namely, "If equals be added to equals, the sums will be equal;" for transposition is neither more nor less than adding equals to equals: thus in ex. 1. there is given

$$4+5-2=6+3+7-9$$

Add to both sides . . . . $+2=$ . . . . $+2$

the sum is . . . . $4+5=6+3+7-9+2$, as in the second step.

Again, add to both sides . . $-5=$ . . . . -5

the sum is . . . . $4 \quad =6+3+7-9+2-5$, as in the third step.

Again, add to both sides $-6=-6$

the sum is . . . . $4-6=3+7-9+2-5$, as in the fourth step.

And so on throughout the operation: whence it appears, that transposition is equivalent to adding the quantity to be transposed, with a contrary sign, to both sides of the equation; and consequently that the quantities resulting from this addition are equal.

Explanation.

In the first step we observe that the -2, being transposed, becomes $+2$; in the second step $+5$, being transposed, becomes -5, &c. it must be likewise observed, that transposing does not affect the equality; the quantities on one side the $=$ being equal to those on the other, as well after transposing, as before: thus, in the given equation, the aggregate of the numbers on each side the $=$ is 7; in the first step it is 9, in the second 4, in the third -2, in the fourth -5, in the fifth -12, in the sixth -3, in the seventh -5, and in the eighth 0 nothing; in every step the sum of the numbers on one side the $=$, incorporated together according to their signs, is equal to that of the numbers so incorporated on the other.

75. *When several quantities are to be transposed,* it is not necessary to take them one at a time, as in the foregoing operation; they may be transposed all together, observing to change the sign of each of them.

2. Given $a+b-c=d$, to transpose $b-c$[d].
 Thus $a=d-b+c$, *the answer.*

3. Given $x=b-z+y-4$, to transpose $-z+y-4$.
 Thus $x+z-y+4=b$, *the answer.*

4. Given $a+b-c=3-y$, to transpose $+b-c$ and $-y$.
 Thus, to transpose $+b-c$ .... $a=3-y-b+c$.
 And to transpose $-y$ .... $a+y=3-b+c$.

76. *The unknown quantity in an equation being connected with known ones by the sign + or −, to find its value.*

RULE. Transpose all the known quantities which are connected with the unknown one, and (Art. 75.) collect them together into one, according to their signs; the result will be the value of the unknown quantity[e].

---

[d] This operation (as was observed in the preceding note) is equivalent to adding $-b+c$ to both sides: thus to the given equation

$$\begin{array}{rrcl} & a+b-c & = & d \\ \text{add} & -b+c & = & -b+c \\ \hline \text{the sum is} & a & = & d-b+c, \text{ the ans.} \end{array}$$

Again, in ex. 3. to the given equation

$$\begin{array}{rrcl} & x & = & b-z+y-4 \\ \text{add} & +z-y+4 & = & +z-y+4 \\ \hline \text{the sum is} & x+z-y+4 & = & b \end{array}$$

and the same of other examples.

[e] In the foregoing rules we are taught *how* transposition is performed, here we learn *its object;* namely, *to get the unknown quantity by itself on one side of the equation.* In the operation of ex. 5. we have $x=4-1+2-3$; now $4-1+2-3$ equals $6-4$, and $6-4$ equals 2, therefore x equals 2, because "things that are equal to the same are evidently equal to one another."

5. Given $x+1-2+3=4$, to find the value of x.

OPERATION.

$$x=4-1+2-3=6-4=2. \text{ *Answer.*}$$

Explanation.

Having transposed $+1-2+3$, they become on the opposite side $-1+2-3$; whence $x=4-1+2-3$, where $+4$ and $+2$ added together give $+6$, and -1 and -3 give -4: whence $6-4$, or 2, is the answer, or value of x.

6. Given $x-1-8+2=3$, to find the value of x.

Thus, by transposing $-1-8+2$, *we have*

$x=3+1+8-2=10$, *the answer* [f].

7. Given $x+5=6$, to find x. *Ans.* $x=1$.

8. Given $x-5=6$, to find x. *Ans.* $x=11$.

9. Given $x+1-2=3$, to find x. *Ans.* $x=4$.

10. Given $x-8+7-6=5$, to find x. *Ans.* $x=12$.

77. *When the unknown quantity is negative.*

RULE. Transpose the unknown quantity, and proceed as before; or if, after the known quantities are transposed, the unknown one be negative, change the signs of all the terms of the equation [g].

---

[f] To prove that the number found by the operation is really the value of the unknown quantity, and that no other number can possibly be its value, this is the rule.

Substitute the value found instead of the unknown quantity in the equation proposed, and if by the adding, subtracting, &c. of the other quantities connected, according to the import of their signs, the result come out the same as the given one, the number found is the value of the unknown quantity: thus, ex. 6. If the number found, viz. 10, be substituted instead of x in the given equation $x-1-8+2=3$, it will become $10-1-8+2=3$. Also ex. 7. If 1 be substituted for x, the given equation becomes $1+5=6$; and it may be remarked, that if any number greater or less than the proper answer be substituted, it will give a result different from that proposed in the equation, and consequently shew that the number substituted is not the true answer. *See note* [i] *at the bottom of the following page.*

[g] It is usual to retain the unknown quantity on the left side of the equation, and to transpose all the known quantities to the right; and then, if the unknown quantity happen to be negative, the sign of every term of the equation is to be changed, in order to make it affirmative.

This change of signs is equivalent to transposing all the terms of the equation, and therefore depends on the same reasons as Art. 74. Otherwise,

11. Given $6=14-x$, to find the value of x.

Thus, by transposing $-x$ .... $6+x=14$.

And by transposing $+6$ .... $x=14-6=8$, *the ans.*

Or, by changing the signs of all the terms of the given equation, we shall have $-6=-14+x$; *whence transposing* -14, *we have* $-6+14=x$, or $8=x$, *as before.*

12. Given $4=5-x$, to find x. *Ans.* $x=1$.
13. Given $2-x=3$, to find x. *Ans.* $x=-1$.
14. Given $10-x=11-5$, to find x. *Ans.* $x=4$.

78. *When the unknown quantity has a coefficient or multiplier.*

RULE I. Transpose all the known quantities to the opposite side, and collect them together as in the former rules.

II. Take away the coefficient from the unknown quantity, and divide the sum of the known ones by it; the quotient will be the value of the unknown quantity [h].

15. Given $3x-6=9$, to find the value of x.

By transposing -6 *we have* $3x=(9+6=)$ 15.

Now, by taking away the coefficient 3, *and dividing* 15 *by it, we have* $x=(\frac{15}{3}=)$ 5 [i], *the answer required.*

When the unknown quantity is found in two terms, and on opposite sides of the equation, if you would make the unknown quantity affirmative, observe, that,

1. When both terms containing the unknown quantity are affirmative, *the least must be transposed.*

2. When both terms are negative, *the greatest must be transposed.*

3. When one term is affirmative, and the other negative, *the negative term must be transposed.*

[h] "If equals be divided by equals, the quotients will be equal;" this self-evident truth is the foundation of the rule; for taking away the coefficient from the unknown quantity, is the same as dividing the term which contains it, by that coefficient; wherefore, taking away the said coefficient, and dividing the rest of the equation by it, is dividing equals by *the same;* consequently the results will be equal.

[i] *Proof of ex.* 15. It is affirmed, that $x=5$

for then $3x=3\times5=15$

and $3x-6=15-6=9$, as in the example. If any number less than 5 be put for x, suppose 4, then the result will be $3\times4-6=12$

16. Given $4y+5-6=7$, to find y.

By transposing $+5-6$, *we have* $4y=(7-5+6=)\ 8$.

By dividing by 4, *we have* $y=(\frac{8}{4}=)\ 2$ [k], *the answer*.

17. Given $2x+10=12$, to find x. *Ans.* $x=1$.

18. Given $4x+9=21$, to find x. *Ans.* $x=3$.

19. Given $12z-3=8$, to find z. *Ans.* $z=\frac{11}{12}$.

20. Given $5y-6+4=11$, to find y. *Ans.* $y=2\frac{3}{5}$.

21. Given $21x+22=50$, to find x. *Ans.* $x=1\frac{1}{3}$.

79. *When the unknown quantity has a divisor.*

RULE I. Transpose all the known quantities to the opposite side, and collect them together as before.

II. Take away the divisor, and multiply the sum of the known quantities by it; the product will be the value of the unknown quantity [l].

22. Given $\frac{x}{3}+4-5=7$, to find the value of x.

By transposing $+4-5$, *we get* $\frac{x}{3}=(7-4+5=)\ 8$.

Then taking away the divisor 3, *and multiplying* 8 *by it, we shall have* $x=(3\times 8=)\ 24$, *the answer* [m].

$-6=6$, too little. Let a greater number, as 6, be proposed; then $3\times 6-6=18-6=12$, too much: for the result proposed is 9; wherefore neither 4 nor 6 can equal x, since one gives a result too little, and the other too much.

[k] *Proof of ex.* 16. If 2 be substituted for y, the given equation becomes $(2\times 4+5-6$, or $8+5-6$, or$)\ 13-6=7$, as was proposed.

[l] This rule depends on a well known self-evident principle, namely, "If equals be multiplied by equals, the products will be equal." Now taking away the divisor from the unknown quantity, is the same as multiplying the term containing it, by that divisor; then, if the divisor be taken away, and all the rest of the equation be multiplied by it, equals will be multiplied by *the same*, and consequently the results will be equal.

[m] *Proof of ex.* 22. By substituting 24 for x in the given equation, it becomes $(\frac{24}{3}+4-5=8+4-5=)\ 12-5=7$. Ex. 23. By substituting -12 for y, the equation becomes $(\frac{-12}{12}+2=)\ -1+2=1$, as required.

34. Given $\sqrt{x}+5=9$, to find the value of x.

By transposition $\sqrt{x}=(9-5=)4$.

By involution $x=(4^2=)\ 16$ [r], *the answer.*

35. Given $\frac{2^3\sqrt{x}}{3}-4=5$, to find x.

By transposition $\frac{2^3\sqrt{x}}{3}=(5+4=)\ 9$.

By multiplication $2^3\sqrt{x}=(3\times 9=)\ 27$.

By division $^3\sqrt{x}=(\frac{27}{2}=)\ 13,5$.

By involution $x=(\overline{13,\ 5}|^3=)\ 2460,375$ [s], *the answer.*

36. Given $\sqrt{z}+15=22$, to find z. *Ans.* $z=49$.

37. Given $3^3\sqrt{y}-6=3$, to find y. *Ans.* $y=27$.

38. Given $\frac{4\sqrt{x}}{5}+2=6$, to find x. *Ans.* $x=25$.

39. Given $\frac{12^4\sqrt{z}}{13}-20+13=5$, to find z. *Ans.* $z=28561$.

82. *When there are known quantities under the same radical sign with the unknown one.*

RULE I. Transpose all the known quantities which are not under the radical sign, as before; and if there be a multiplier or divisor not under it, they must be taken away.

II. Involve both sides to the power implied by the radical sign, whereby it will be taken away.

III. Transpose, multiply, or divide afterwards, as the case may require, and the result will be the answer [t].

---

[r] *Proof* $(\sqrt{16}+5=)\ 4+5=9$, as required.

[s] *Proof* $(\frac{2\times{}^3\sqrt{2460.375}}{3}-4=\frac{2\times 13.5}{3}-4=\frac{27}{3}-4=)\ 9-4=5$, as required.

[t] This rule is evidently a compound of all the preceding ones, and is meant to shew in what order they are to be applied in disentangling the unknown quantity. We follow the same maxim here that women do in clearing a tangled skein; we clear the *outside* tangle first, and proceed in order, always taking the *outside* one, until at last we arrive at the unknown quantity, after having disengaged it from a complication of multipliers, divisors, radical signs, &c. always proceeding in the *reverse* order to that in which the complication is made.

40. Given $\sqrt{3x+4}-8+20=16$, to find the value of x.

By transposition $\sqrt{3x+4}=(16+8-20=)\ 4$.

By involution $3x+4=(\overline{4}|^2=)\ 16$.

By transposition $3x=(16-4=)\ 12$.

By division $x=(\frac{12}{3}=)\ 4$ [w], *the answer.*

41. Given $\frac{2\sqrt[3]{7z-6}}{3}+\frac{3}{4}=\frac{25}{12}$, to find z.

By transposition $\frac{2\sqrt[3]{7z-6}}{3}=(\frac{25}{12}-\frac{3}{4}=\frac{25-9}{12}=\frac{16}{12}=)\frac{4}{3}$.

By multiplication $2\sqrt[3]{7z-6}=(\frac{4}{3}\times3=)\ 4$.

By division $\sqrt[3]{7z-6}=(\frac{4}{2}=)\ 2$.

By involution $7z-6=(\overline{2}|^3=)\ 8$.

By transposition $7z=(8+6=)\ 14$.

By division $z=(\frac{14}{7}=)2$ [x], *the answer.*

42. Given $\sqrt{2y-1}=5$, to find y. *Ans.* $y=13$.

43. Given $\sqrt{11x+11}-10=1$, to find x. *Ans.* $x=10$.

44. Given $4\sqrt[3]{2z-16}+5=21$, to find z. *Ans.* $z=40$.

45. Given $\frac{7\sqrt[4]{9x-27}}{8}-2=\frac{5}{8}$, to find x. *Ans.* $x=12$.

83. *When the unknown quantity is in any power.*

RULE I. Transpose, multiply, divide, &c. as before, until the power of the unknown quantity remains alone on one side of the equation.

II. Extract the root implied by the index of the unknown quantity from both sides, and the result will be the answer [y].

---

[w] *Proof.* $(\sqrt{3\times4+4}-8+20=\sqrt{16}+12=)\ 4+12=16$.

[x] *Proof.* $(\frac{2\times\sqrt[3]{7\times2-6}}{3}+\frac{3}{4}=\frac{2\times\sqrt[3]{8}}{3}+\frac{3}{4}=\frac{2\times2}{3}+\frac{3}{4}=)\frac{4}{3}+\frac{3}{4}=\frac{16+9}{12}=\frac{25}{12}$, as proposed.

[y] It has been shewn, that like powers of equal quantities are equal; whence it follows, that their like roots will be equal: or it may be proved thus; let

46. Given $\frac{3x^2}{4}+5=80$, to find the value of x.

By transposition $\frac{3x^2}{4}=(80-5=)\ 75$.

By multiplication $3x^2=(4\times 75=)\ 300$.

By division $x^2=(\frac{300}{3}=)\ 100$.

By evolution $x=(\sqrt{100}=)\ 10$, *the answer.*

47. Given $\frac{5x^3+5}{8}-200=15$, to find x.

By transposition $\frac{5x^3+5}{8}=(15+200=)\ 215$.

By multiplication $5x^3+5=(8\times 215=)\ 1720$.

By transposition $5x^3=(1720-5=)\ 1715$.

By division $x^3=(\frac{1715}{5}=)\ 343$.

By evolution $x=(\sqrt[3]{343}=)\ 7$, *the answer.*

48. Given $3y^2-40=68$, to find y. *Ans.* $y=6$.

49. Given $12z^3+32=800$, to find z. *Ans.* $z=4$.

50. Given $\frac{2x^4}{3}+100-24=130$, to find x. *Ans.* $x=3$.

51. Given $\frac{4x^2-5}{6}+7=8$, to find x. *Ans.* $x=1.6583$, &c.

84. *When the unknown quantity is in the form of a fraction, in two or more terms of the equation.*

RULE I. Multiply each term of the equation successively by every denominator except its own, whereby all the denominators will be taken away; this is called *clearing the equation of fractions.*

$a=b$; then, if $\sqrt{a}$ be not equal to $\sqrt{b}$, one of them is the greater; let this be $\sqrt{a}$; then, because $\sqrt{a}$ is greater than $\sqrt{b}$, $\sqrt{a}\times\sqrt{a}$ $(=a)$ is greater than $\sqrt{b}\times\sqrt{b}$, $(=b,)$ that is, a is greater than b; but they are equal by hypothesis, which is absurd; wherefore $\sqrt{a}$ is not greater than $\sqrt{b}$: and in the same manner it may be shewn, that it is not less; wherefore they are equal: and the same may be proved of any other like roots of these, or of any two equal quantities.

II. Reduce the resulting equation, as the case may require, by the foregoing rules, for the answer [a].

52. Given $\frac{x}{2}+\frac{x}{3}=7+\frac{x}{4}$, to find x.

By multiplying by 2, *we have* $x+\frac{2x}{3}=14+\frac{x}{2}$.

By multiplying by 3 ...... $3x+2x=42+\frac{3x}{2}$.

By multiplying by 2 ...... $6x+4x=84+3x$.

By transposition .... $(6x+4x-3x=)\ 7x=84$.

By division .... $x=(\frac{84}{7}=)$ 12, *the answer.*

53. Given $\frac{2x+3}{5}+x=37-\frac{7x}{8}$, to find x.

Multiplying by 5, *we have* $2x+3+5x=185-\frac{35x}{8}$.

Multiplying by 8 ...... $16x+24+40x=1480-35x$.

By transposition ...... $(16x+40x+35x=)91x=(1480-24=)$ 1456.

And by division, $x=(\frac{1456}{91}=)$ 16, *the answer.*

85. This operation may be more expeditiously performed by taking any common multiple of all the denominators, (the least is the most convenient,) dividing it by each denominator separately, multiplying the quotient by the numerator, and making the result the coefficient of the term from which it is derived [a].

---

[a] This rule requires only the successive multiplication of equals by the same, and therefore the resulting products will be equal, as is manifest from what has been shewn in the note on Art. 78.

[a] This rule depends on the axiom we have frequently quoted, viz. "if equals be multiplied by the same, the products will be equal;" now here it is proposed to multiply both sides of the equation (or equals) by the least common multiple of all the denominators, and this multiplication is evidently performed by dividing in each instance the common multiple by the denominator, and multiplying the quotient by the numerator: for let ex. 54. be proposed, where $\frac{x}{3}-\frac{x}{4}=\frac{x}{6}-12$; multiply every term of this equation by the least common

54. Given $\frac{x}{3}-\frac{x}{4}=\frac{x}{6}-12$, to find the value of x.

The least common multiple of 3, 4, and 6, is 12.

Wherefore $\frac{12}{3}=4$ *the first coefficient,* $\frac{12}{4}=3$ *the second,* $\frac{12}{6}=2$ *the third, and* $\frac{12}{1}=12$ *the fourth; wherefore if the several denominators be taken away, and the first x be multiplied by 4, the second by 3, the third by 2, and the* -12 *by 12, we shall have*

$4x-3x=2x-144$. *And by transposition,*

$144=2x+3x-4x$, *that is* $x=144$, *the answer.*

55. Given $\frac{2z}{5}+4=\frac{5z}{7}-1$, *to find z.*

The least common multiple of 5 and 7 is 35.

Whence the multipliers are $\frac{35}{5}=7$, *and* $\frac{35}{7}=5$, *and 35.*

Therefore $14z+140=25z-35$; *which by transposition becomes* $11z=175$, *whence* $z=(\frac{175}{11}=)15\frac{10}{11}$, *the answer.*

56. Given $\frac{x}{2}+\frac{x}{3}=5$, to find x. *Ans.* $x=6$.

57. Given $\frac{y}{4}-\frac{y}{6}=2$, required y? *Ans.* $y=24$.

58. Given $\frac{2z}{3}-\frac{z}{2}=\frac{z+2}{8}+1$, to find z. *Ans.* $z=30$.

59. Given $4z+\frac{5z}{6}=\frac{108-2z}{7}+46$, to find z. *Ans.* $z=12$.

86. *To turn an analogy into an equation.*

RULE. Multiply the two extreme terms together, and also the two mean terms together, and make the former product equal to the latter; then reduce the equation as before [b].

---

multiple 12, and there arises $\frac{12x}{3}-\frac{12x}{4}=\frac{12x}{6}-144$; but $\frac{12}{3}=4, \frac{12}{4}=3, \frac{12}{6}=2$, and $\frac{12}{1}=12$; therefore our equation becomes $4x-3x=2x-144$, as in the example; whence the rule is evident.

[b] If four numbers are proportionals, the product of the extremes is equal to

60. If $x : 12 :: 2 : 3$, required the value of x?

Thus ($x \times 3 = 12 \times 2$, *or*) $3x = 24$, *whence by division* $x = (\frac{24}{3} =) 8$, *the answer.*

61. If $z - 2 : 20 :: 2 : 5$, required z?

Thus ($\overline{z-2} \times 5 = 20 \times 2$, *or*) $5z - 10 = 40$, *whence* $5z = 50$, *and* $z = 10$, *the answer.*

62. Given $\frac{y+3}{5} : 5 :: \frac{1}{5} : \frac{1}{2}$, to find y.

Thus ($\frac{y+3}{5} \times \frac{1}{2} = 5 \times \frac{1}{5}$, *or*) $\frac{y+3}{10} = 1$, *whence* $y + 3 = 10$, *and* $y = 7$, *the answer.*

63. Given $2z : 3 :: 3 : 2$, to find z. *Ans.* $z = 2\frac{1}{4}$.

64. Given $2 + y : 16 :: 7 : 8$, to find y. *Ans.* $y = 12$.

65. Given $\frac{x}{3} : \frac{1}{4} :: \frac{5}{7} : \frac{4}{9}$, to find x. *Ans.* $x = 1\frac{23}{112}$.

66. Given $\frac{x}{2} + \frac{x}{3} : x - 2 :: 10 : 11$, to find x. *Ans.* $x = 24$.

87. When the same quantity, with *the same sign,* is on both sides of the equation, it may be taken away from both; and if every term of an equation be multiplied by any (the same) quantity, or divided by any (the same) quantity, such multiplier, or divisor, may be taken away from them all[c].

67. Given $4x - \frac{2x}{3} = \frac{7}{8} - \frac{2x}{3}$, to find the value of x.

Leaving out the common quantity $-\frac{2x}{3}$, *we shall have* $4x = \frac{7}{8}$, *whence by division* $x = (\frac{7}{8} \div 4 =) \frac{7}{32}$, *the answer.*

68. Given $3az + \frac{ab}{2} = 2ab - 3a$, to find z.

---

the product of the means; this will be obvious from any example of four numbers, which are known to be proportionals:

Thus if $2 : 4 :: 3 : 6$, then will $2 \times 6 = 4 \times 3$.

And if $12 : 4 :: 21 : 7$, then will $12 \times 7 = 4 \times 21$.

And the same will appear in every instance.

[c] This is merely taking equals from equals, dividing equals by equals, or multiplying equals by equals; wherefore in each case the results will be equal.

Leaving out the common multiplier a, we shall have $3z+\frac{b}{2}=2b-3$, *which by transposition becomes* $3z=2b-3-\frac{b}{2}$, *and by division* $z=(\frac{2b}{3}-1-\frac{b}{6})=\frac{b}{2}-1$, *the answer.*

69. Given $\frac{2x}{5}-\frac{3}{5}+\frac{x^2}{a}=\frac{4}{5}+\frac{x^2}{a}$, to find x.

Leaving out $\frac{x^2}{a}$, *we have* $\frac{2x}{5}-\frac{3}{5}=\frac{4}{5}$, *and leaving out the common divisor 5, we have* $2x-3=4$, *whence by transposition* $2x=7$, *and by division* $x=(\frac{7}{2}=)3\frac{1}{2}$, *the answer.*

70. Given $2x+3y=x+3y+8$, to find x. *Ans.* $x=8$.

71. Given $3y+3=3y+5x$, to find x. *Ans.* $x=\frac{3}{5}$.

72. Given $\frac{x+3}{5}-\frac{x}{4}=\frac{4}{5}-\frac{x}{4}$, to find x. *Ans.* $x=1$.

88. Promiscuous Examples for Practice.

1. Given $6x-5=3x+4$, to find x.

By Art. 75. $3x=9$.

By Art. 78. $x=3$, *the answer.*

2. Given $21-2x=2x-21$, to find x.

By Art. 75. $4x=42$.

By Art. 78. $x=10\frac{1}{2}$, *the answer.*

3. Given $\frac{22}{x+3}=2$, to find x.

By Art. 79. $22=2x+6$.

By Art. 75. $2x=16$.

By Art. 78. $x=8$, *the answer.*

4. Given $\sqrt{\frac{3x}{2}} : \sqrt{x-1} :: 3 : 1$, to find x.

By Art. 86. $\sqrt{\frac{3x}{2}}=3\sqrt{x-1}$.

By Art. 81. $\frac{3x}{2}=(9\times\overline{x-1}=)9x-9$.

By Art. 79. $3x=18x-18$.

By Art. 75. $15x=18$.

By Art. 78. $x=1\frac{1}{5}$, *the answer.*

5. Given $\frac{z-3}{2}+\frac{z}{3}=10-\frac{z-5}{4}$, to find z.

By Art. 85. $6z-18+4z=120-3z+15$.
By Art. 75. $13z=153$.
By Art. 78. $z=11\frac{10}{13}$, *the answer.*

6. Given $\frac{45}{2x+3}=\frac{57}{4x-5}$, to find x.

By Art. 79. $180x-225=114x+171$.
By Art. 75. $66x=396$.
By Art. 78. $x=6$, *the answer.*

7. Given $y+\sqrt{4+y^2}=\frac{8}{\sqrt{4+y^2}}$, to find y.

By Art. 79. $y\sqrt{4+y^2}+4+y^2=8$.
By Art. 75. $y\sqrt{4+y^2}=4-y^2$.
By Art. 81. $4y^2+y^4=16-8y^2+y^4$.
By Art. 87. $4y^2=16-8y^2$.
By Art. 75. $12y^2=16$.
By Art. 78. $y^2=\frac{4}{3}$.
By Art. 83. $y=\frac{2}{\sqrt{3}}=1.1547$, *&c. the answer.*

8. Given $13-\sqrt{3x}=\sqrt{13+3x}$, to find x.

By Art. 81....... $169-26\sqrt{3x}+3x=13+3x$.
By Art. 87....... $169-26\sqrt{3x}=13$.
By Art. 77....... $26\sqrt{3x}=156$.
By Art. 78......... $\sqrt{3x}=6$.
By Art. 81........... $3x=36$.
By Art. 78............ $x=12$, *the answer.*

9. Given $\sqrt{x}+\sqrt{a+x}=\frac{1}{\sqrt{a+x}}$, to find x.

By Art. 79....... $\sqrt{ax+x^2}+a+x=1$.
By Art. 75....... $\sqrt{ax+x^2}=1-a-x$.
By Art. 81......... $ax+x^2=1-2a-2x+a^2+2ax+x^2$.
By Art. 87......... $ax=1-2a-2x+a^2+2ax$.
By Art. 75......... $2x-ax=1-2a+a^2$.
By Art. 78......... $x=\frac{1-2a+a^2}{2-a}$, *the answer.*

10. Given $\frac{x}{2}+\frac{x}{3}=39-\frac{x}{4}$, to find x. *Ans.* $x=36$.

11. Given $\frac{x+1}{3}+\frac{x+3}{4}=\frac{x+4}{5}+16$, to find x. *Ans.* $x=41$.

12. Given $\frac{2y-3}{4}-\frac{y}{7}=\frac{4y-4}{5}-3\frac{69}{140}$, to find y. *Ans.* $y=8$.

13. Given $\frac{2\sqrt{5z}}{5}+16=20$, to find z. *Ans.* $z=20$.

14. Given $\frac{6}{\sqrt{2x+8}}=\frac{7}{\sqrt{3x+7}}$, to find x. *Ans.* $x=14$.

15. Given $\frac{35z}{z-3}=\frac{42z}{z-2}$, to find z. *Ans.* $z=8$.

16. Given $\frac{2x^3}{3}+\frac{3x}{4}=\frac{4x^3}{5}-\frac{2x}{3}$, to find x. *Ans.* $x=3.259$, &c.

17. Given $3\times\overline{z-2}=\frac{z+2}{3}+2z$, to find z. *Ans.* $z=10$.

18. Given $\overline{b^4+x^4}|^{\frac{1}{4}}=\overline{a^2+x^2}|^{\frac{1}{2}}$, to find x. *Ans.* $x=\frac{1}{a}\sqrt{\frac{b^4-a^4}{2}}$.

89. REDUCTION OF SIMPLE EQUATIONS, INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

When there are two or more unknown quantities, whose values are required, in order to obtain determinate answers, it is necessary that there should be as many equations (independent of each other) given, as there are unknown quantities to be found [d]; thus, when there are two unknown quantities, two equations must be proposed; for three unknown quantities, there must be three equations; for four, four equations, &c.

---

[d] If there are more equations than unknown quantities, the superfluous equations will either coincide with some of the others, or contradict them; wherefore, in the former case they are unnecessary, and in the latter, detrimental; rendering the proposed solution impossible.

If there are fewer equations than unknown quantities, the problem will admit of many answers; thus, let $x+y=4$, here is but one equation, and two unknown quantities to be found; now x may equal 3, then $y=1$; if $x=2$, then $y=2$; if $x=1$, then $y=3$; wherefore, in this instance, both x and y admit of three interpretations, using whole numbers only; if fractions be admitted, the values of x and y will be innumerable.

In all these cases there is one general mode of procedure, namely, we *exterminate* all the unknown quantities from the operation, except *one*, the value of which is to be found by the foregoing methods; having obtained the value of *this*, the value of each of the other unknown quantities will be readily found by means of it, from some of the preceding equations.

For two unknown quantities.

90. *First Method.*

RULE I. Find the value of one unknown quantity in each of the equations, by the foregoing rules; it may be either of the two quantities at pleasure, but must be the same unknown quantity in both equations.

II. Put the two values thus found equal to each other *; this equation will then contain but one unknown quantity, the value of which is to be found by the preceding rules.

III. Having thus found the value of one unknown quantity, substitute it for that quantity in either of the preceding equations, and the value of the other unknown quantity will be found.

EXAMPLES.

1. Given $x+y=13$, and $x-y=3$, to find the value of x and y.

First, to find the value of x in each equation.

From the first, $x=13-y$; *and from the second,* $x=3+y$; *these two values of x are evidently equal to each other; wherefore* $13-y=3+y$, *whence* $2y=10$, *and* $y=5$; *now substitute this value of y in either of the former equations, suppose in* $x=3+y$, *and it becomes* $x=(3+y=)\ 3+5=8$.

Wherefore $x=8$, *and* $y=5$.

2. Given $2x+3y=17$, and $3x-2y=6$, to find x and y.

From the first equation, $2x=17-3y$,

$$\textit{And}\ \ x=\frac{17-3y}{2}.$$

From the second equation, $3x=6+2y$,

$$\textit{And}\ \ x=\frac{6+2y}{3}.$$

* These two values being equal to *the same* unknown quantity, are evidently equal to one another: the unknown quantity, whose two values (or rather, two different expressions of the *same* value) are thus found, is said to be *exterminated*, because it does not appear in the resulting equation.

Making these two values of x equal to each other, we shall have $\frac{17-3y}{2}=\frac{6+2y}{3}$*: this equation cleared of fractions,* (Art. 79.) *becomes* $51-9y=12+4y$.

Whence $13y=39$, *and* $y=(\frac{39}{13}=)\ 3$.

And $x=(\frac{6+2y}{3}=)\frac{6+6}{3}=\frac{12}{3}=4$.

3. Given $4x-y=26$, and $\frac{x}{2}-\frac{y}{3}=2$, to find x and y.

From eq. 1. $x=\frac{26+y}{4}$. *From eq.* 2. $x=4+\frac{2y}{3}$.

Whence $\frac{26+y}{4}=4+\frac{2y}{3}$.

Then $78+3y=48+8y$.

Whence $5y=30$, *and* $y=6$.

Wherefore $x=(\frac{26+y}{4}=)\ \frac{26+6}{4}=\frac{32}{4}=8$.

4. Given $x+4y=18$, and $x-3y=4$, to find x and y. *Ans.* $x=10$, $y=2$.

5. Given $4x+3y=25$, and $5x-4y=8$, to find x and y. *Ans.* $x=4$, $y=3$.

6. Given $\frac{x+y}{2}=10$, and $\frac{2x+3y}{4}=12$, to find x and y. *Ans.* $x=12$, $y=8$.

7. Given $\frac{x}{2}+\frac{2y}{3}=20$, and $\frac{3x}{4}-\frac{y}{5}=12$, to find x and y. *Ans.* $x=20$, $y=15$.

8. Given $\frac{x}{4}+3y=217$, and $\sqrt{x} : \sqrt{y} :: 5 : 4$, to find x and y. *Ans.* $x=100$, $y=64$.

91. *Second Method.*

RULE I. Find the value of either of the unknown quantities in one of the given equations, and substitute this value for that quantity in the other given equation; this equation will then contain only one unknown quantity, which may be found as before [f].

---

[f] This rule is evident, for it is plain that in any expression whatever, we

II. Find the value of the other unknown quantity, as directed in the preceding rule.

9. Given $x+y=20$, and $x-y=8$, to find x and y.

From eq. 2. $x=8+y$; *this value being substituted for* x *in the first equation,* $(x+y=20,)$ *gives* $8+y+y=20$, *or* $2y+8=20$, *whence* $2y=12$ *and* $y=6$; *this value of* y *being substituted for* y *in the equation,* $x=8+y$, *gives* $x=8+6=14$.

10. Given $2x+3y=7$, and $3x-2y=4$, to find x and y.

From eq. 1. $x=\frac{7-3y}{2}$, *whence* $3x=\frac{21-9y}{2}$; *this value substituted for* $3x$ *in the second equation, it becomes* $\frac{21-9y}{2}-2y=4$, *which by multiplication and transposition, becomes* $13y=13$, *whence* $y=1$; *this value being substituted for* y *in the equation* $x=\frac{7-3y}{2}$, *it becomes* $\frac{7-3}{2}=\frac{4}{2}=2$, *for the value of* x.

11. Given $\frac{y+z}{3}+\frac{y-z}{4}=59$, and $y:3z::11:5$, to find y and z.

From the analogy $5y=33z$, *and* $y=\frac{33z}{5}$; *this value of* y *being substituted in the given equation, we have,* $\frac{\frac{33z}{5}+z}{3}+\frac{\frac{33z}{5}-z}{4}=59$; *whence* $\frac{33z+5z}{15}+\frac{33z-5z}{20}=59$, *or* $\frac{38z}{15}+\frac{7z}{5}=59$, *or* $\frac{59z}{15}=59$; *whence* $59z=59\times15$, *or* $z=15$; *this value substituted for* z *in the equation* $y=\frac{33z}{5}$, *it becomes* $y=\frac{33\times15}{5}=33\times3=99$.

12. Given $x+y=10$, and $x-y=2$, to find x and y. *Ans.* $x=6$, $y=4$.

13. Given $2x+3z=38$, and $6x+5z=82$, to find x and z. *Ans.* $x=7$, $z=8$.

14. Given $y-2z=7$, and $2y-z=20$, to find y and z. *Ans.* $y=11$, $z=2$.

---

can substitute their equals, instead of any of the quantities which compose it, without altering the value of that expression.

15. Given $2x+3y=34$, and $x+y : x-y :: 7 : 1$, to find x and y. *Ans.* $x=8$, $y=6$.

92. *Third Method.*

RULE I. Multiply the first equation by the coefficient of one of the quantities in the second, and the second equation by the coefficient of the same quantity, in the first; the products will be two new equations, in both which the coefficients of that quantity will be the same.

II. If the terms, with equal coefficients, have like signs, subtract one of the new equations from the other; but if they have unlike signs, add the new equations together: the result will be an equation with only one unknown quantity, which may be found as before [s].

16. Given $2x+5y=23$, and $7x-3y=19$, to find x and y.

To exterminate x, multiply the first equation by 7, and the second by 2, and the products (or new equations) will be

$$\left.\begin{array}{r} 14x+35y=161. \\ \textit{And}\ \ 14x-\ \ 6y=\ \ 38. \end{array}\right\}\ \textit{New equations.}$$

Whence $41y=123$ *by subtracting the lower from the upper, whence* $y=(\frac{123}{41}=)\ 3$.

Now to exterminate y, multiply the first given equation by 3, and the second by 5, and the new equations will be

$$\left.\begin{array}{r} 6x+15y=69. \\ \textit{And}\ \ 35x-15y=95. \end{array}\right\}\ \textit{New equations.}$$

Whence $41x\ =\ 164$ *(by adding) and* $x=(\frac{164}{41}=)\ 4$.

17. Given $\frac{2x}{3}+y=24$, and $\frac{4x}{5}-\frac{y}{2}=1\frac{3}{5}$, to find x and y.

---

[s] We are at liberty to employ any process, where *equals operate in a similar manner upon equals*: under this restriction, we are authorised to make use of addition, subtraction, multiplication, division, involution, and evolution, according as it suits our purpose; in this rule equal multiplication is used, but sometimes equal division, when it can be used, makes the work shorter.

Multiply eq. 1. *by* $\frac{4}{5}$, *and eq.* 2. *by* $\frac{2}{3}$, *and the new equations are*

$$\frac{8x}{15}+\frac{4y}{5}=\frac{96}{5}.$$

$$\text{And } \frac{8x}{15}-\frac{y}{3}=(\frac{2}{3}\times 1\frac{3}{5}=\frac{2}{3}\times\frac{8}{5}=)\ \frac{16}{15}.$$

$$\text{By subtraction } \frac{4y}{5}+\frac{y}{3}=\frac{96}{5}-\frac{16}{15}=(\frac{288-16}{15}=)\frac{272}{15}.$$

This equation multiplied by (the least common multiple of its denominators, viz.) 15, *gives* $(12y+5y=)\ 17y=272$; *whence* $y=(\frac{272}{17}=)\ 16$; *this value substituted for y in the first given equation, we have* $(\frac{2x}{3}+y=)\ \frac{2x}{3}+16=24$, *or* $\frac{2x}{3}=8$; *wherefore* $2x=24$, *and* $x=12$.

18. Given $x+y=1$, and $x-y=\frac{1}{3}$, to find x and y.

Here multiplication is unnecessary; therefore by adding both equations together, we get $2x=(1\frac{1}{3}=)\ \frac{4}{3}$, *whence* $x=\frac{2}{3}$; *and by subtracting the second from the first,* $2y=(1-\frac{1}{3}=)\ \frac{2}{3}$, *whence* $y=\frac{1}{3}$.

19. Given $2x+3y=13$, and $5x-2y=4$, to find x and y. *Ans.* $x=2$, $y=3$.

20. Given $x+y=5$, and $x-y=1$, to find x and y. *Answer* $x=3$, $y=2$.

21. Given $7y+9z=169$, and $8z-9y=-2$, to find y and z. *Ans.* $y=10$, $z=11$.

22. Given $2x+9=5z+8$, and $3z-x=x+\frac{1}{5}$, to find x and z. *Ans.* $x=\frac{1}{2}$, $z=\frac{2}{5}$.

For three unknown quantities.

93. *First Method.*

RULE I. Let x, y, and z, be the three unknown quantities, whose values are sought; first find the value of x in each of the three given equations.

II. Make the value of x in the first equation equal to the value of x in the second, and a new equation will be formed, involving only y and z, with known quantities.

III. Make the value of x in the first equation equal to the value of x in the third, and a second new equation will be formed, involving in like manner only y, z, and known quantities [h].

IV. Find the value of y and z in these two new equations, by either of the former methods; then, by substituting these values for y and z respectively, in either of the given equations, the value of x will be readily obtained.

23. Given $x+y+z=9$, $x+3y-3z=7$, and $x-4y+8z=8$; required the values of x, y, and z?

From the first equation $x=9-y-z$.
From the second ..... $x=7-3y+3z$.
From the third ..... $x=8+4y-8z$.

New equations $\begin{cases} 9-y-z=7-3y+3z. \\ 9-y-z=8+4y-8z. \end{cases}$

From the first new eq. $2y=4z-2$, *whence* $y=2z-1$.

From the second new eq. $5y=7z+1$, *whence* $y=\dfrac{7z+1}{5}$.

By making these two values of y *equal, we have* $2z-1=\dfrac{7z+1}{5}$, *whence* $10z-5=7z+1$, *or* $3z=6$, *and* $z=2$; *also* $y=(2z-1=)\ 3$, *and* $x=(9-y-z$, *as above* $=)\ 9-3-2=4$.

24. Given $2x+3y+4z=20$, $3x-4y+5z=10$, and $4x+5y-z=11$, to find x, y, and z.

From the first eq. $x=\dfrac{20-3y-4z}{2}$.

From the second $x=\dfrac{10+4y-5z}{3}$.

From the third $x=\dfrac{11-5y+z}{4}$.

---

[h] From an attentive consideration of the rule it will appear, that the object proposed is to exterminate one and the same unknown quantity from two of the given equations; when this is effected, we have two new equations only, involving two unknown quantities only: the subsequent part of the operation will therefore depend on the rules for two unknown quantities.

New equations $\begin{cases} \dfrac{20-3y-4z}{2}=\dfrac{10+4y-5z}{3}. \\ \dfrac{20-3y-4z}{2}=\dfrac{11-5y+\ z}{4}. \end{cases}$

From the first new eq. $60-9y-12z=20+8y-10z$, *or* $y=\dfrac{40-2z}{17}$.

From the second, $80-12y-16z=22-10y+2z$, *or* $y=29-9z$.

Wherefore $\dfrac{40-2z}{17}=29-9z$, *or* $40-2z=493-153z$, *whence* $z=3$; *also* $y=(29-9z=)\ 2$, *and* $x=(\dfrac{20-3y-4z}{2}=)\ 1$.

25. Given $\dfrac{x+y+z}{2}+x=13$, $\dfrac{x+y}{3}+z=11\frac{1}{3}$, and $x+y+z : x+y-z :: 9 : 1$, to find x, y, and z.

From the first eq. $x=\dfrac{26-y-z}{3}$.

From the second $x=34-y-3z$.

From the analogy $x=\dfrac{5z-4y}{4}$.

Wherefore $\dfrac{26-y-z}{3}=34-y-3z$, *and* $\dfrac{26-y-z}{3}=\dfrac{5z-4y}{4}$.

These equations reduced as before, give $y=6$, $z=8$, *and* $x=4$.

26. Given $x+y+z=15$, $x+y-z=3$, and $x-y+z=5$, to find x, y, and z. *Ans.* $x=4$, $y=5$, $z=6$.

27. Given $2x-y-z=10$, $3x-2y+z=23$, and $5x-4y-3z=9$, to find x, y, and z. *Ans.* $x=12$, $y=9$, $z=5$.

28. Given $\dfrac{x}{2}+\dfrac{y}{3}+z=13$, $2x-3y+\dfrac{z}{2}=4$, and $x+2y : 2y+z :: 5 : 8$, to find x, y, and z. *Ans.* $x=4$, $y=3$, $z=10$.

94. *Second Method.*

RULE. Find the value of one of the unknown quantities in either of the equations, and substitute it for that quantity in both the remaining equations; these two equations will then contain only two unknown quantities, which may be found as before [i].

---

[i] The foregoing rule is similar to the first method for two unknown quantities, this is similar to the second.

29. Given $x+y+z=15$, $x+y-z=11$, and $x-y-z=5$, to find x, y, and z.

From the first equation $x=15-y-z$; *this value being substituted for* x *in the second and third equations, we shall have* $15-y-z+y-z=11$, *whence* $2z=4$, *and* $z=2$; *and* $15-y-z-y-z=5$, *whence* $2y=10-2z$; *and* $y=5-z=(5-2=)\,3$, *and* $x=15-y-z=(15-3-2=)\,10$.

30. Given $2x+3y+z=40$, $3x+y-z=13$, and $4x+5y=z+46$, to find x, y, and z.

From the third equation $z=4x+5y-46$; *this value substituted for* z *in the first and second equations, gives* $2x+3y+4x+5y-46=40$, *and* $3x+y-4x-5y+46=13$; *whence by reduction* $x=(\frac{86-8y}{6}=)\,\frac{43-4y}{3}$, *and* $x=33-4y$; *whence* $\frac{43-4y}{3}=33-4y$, *whence* $y=7$; *and* $x=33-4y=(33-28=)\,5$, *also* $z=4x+5y-46=(20+35-46=)\,9$.

31. Given $x+y+z=90$, $x-2y+3z=40$, and $x+4y-5z=60$, to find x, y, and z. *Ans.* $x=40$, $y=30$, $z=20$.

32. Given $2x+3y+4z=32$, $4x+3y+z=23$, and $x-y+2z=11$, to find x, y, and z. *Ans.* $x=3$, $y=2$, $z=5$.

95. PROMISCUOUS EXAMPLES FOR PRACTICE.

It frequently happens, that the unknown quantities may be exterminated by methods more simple and easy than any of the foregoing; the application of these must be left to exercise the skill of the operator, as no general rules can be given that will apply to every case.

1. Given $x+3y-4z=10$, $3x+5y+3z=66$, and $5x+2y+7z=80$, to find x, y, and z.

From the sum of the first and third subtract the second, and there remains $3x=24$, *whence* $x=8$; *substitute this value for* x *in the first and second, and* $8+3y-4z=10$, *also* $24+5y+3z=66$; *these two equations reduced, give* $y=6$, *and* $z=4$.

2. Given $\overline{x+2y+z}|^2=144$, $\overline{x+y+z}|^{\frac{1}{2}}=3$, and $\frac{1}{2}\overline{x+y-3z}=\frac{1}{2}$, to find x, y, and z.

From the square root of the first, subtract the square of the second, and there arises $y=3$; *subtract twice the third from*

the square of the second, and $4z=8$, *whence* $z=2$; *substitute these values of* y *and* z *in the square of the second, (viz.* $x+y+z=9$,*) and we shall have* $x=4$.

3. Given $x+\frac{y+z}{2}=19$, $y+\frac{x+z}{3}=15$, and $z+\frac{x+y}{4}=12$, to find x, y, and z.

From three times the second take twice the first, and $2y-x=7$, *or* $x=2y-7$. *From six times the second take twice the first, and* $5y+z=52$, *or* $z=52-5y$. *In four times the third, viz.* $4z+x+y=48$, *substitute the values of* x *and* z *as found above, and it becomes* $208-20y+2y-7+y=48$; *whence* $y=9$, $x=(2y-7=)\ 11$, *and* $z=(52-5y=)\ 7$.

4. Given $xy=6$, and $x^4+y^4=97$, to find x and y.

Add twice the square of the first eq. to, and subtract it from the second, then find the square root of the sum and difference, which will be respectively $x^2+y^2=13$, *and* $x^2-y^2=5$; *by adding these two together, we get* $2x^2=18$, *or* $x^2=9$, *whence* $x=3$; *and by subtracting the latter of them from the former,* $2y^2=8$, *or* $y^2=4$, *whence* $y=2$.

5. Given $x^2-y^2=7$, and $xy=12$, to find x and y.

To the square of the first add four times the square of the second, the square root of the sum will be $x^2+y^2=25$; *add the first equation to, and subtract it from this, and* $2x^2=32$, $x^2=16$, *and* $x=4$; *also* $2y^2=18$, $y^2=9$, *and* $y=3$.

6. Given $\frac{xy}{y+3}=x-50$, and $\frac{xy}{y-2}=x+200$, to find x and y.

Multiply the first by $y+3$, *and the second by* $y-2$, *and we shall have* $xy=xy-50y+3x-150$; *whence* $x=\frac{50y+150}{3}$, *and* $xy=xy+200y-2x-400$, *whence* $x=100y-200$; *therefore* $\frac{50y+150}{3}=100y-200$, *or* $50y+150=300y-600$; *whence* $250y=750$, *and* $y=3$; *also* $x=(100y-200=)\ 100$.

7. Given $\frac{x}{x+y}=\frac{2}{3}$, and $y\times\overline{x+y}=48$, to find x and y.

Multiply both equations together, and $xy=(\frac{2}{3}\times 48=)\ 32$; *but* $(y\times\overline{x+y}=)\ xy+y^2=48$; *subtract the preceding equation from this, and* $y^2=16$, *whence* $y=4$; *this value substituted for* y *in the equation* $xy=32$, *gives* $4x=32$, *whence* $x=8$.

8. Given $x^2+xy=15$, and $x^2-y^2=5$, to find x and y.

Subtract the second from the first, and $xy+y^2=10$; *add this to the first, and* $x^2+2xy+y^2=25$; *extract the square root of this, and* $x+y=5$; *substitute this value for* $x+y$ *in the first equation* $(x^2+xy=)\ \overline{x+y}\times x=15$, *and it becomes* $5x=15$, *whence* $x=3$; *this value substituted for* x *in the equation* $x+y=5$, *gives* $y=2$.

9. Given $\overline{x+y}\times\frac{x}{y}=60$, and $\overline{x+y}\times\frac{y}{x}=2\frac{2}{5}$, to find x and y.

Multiply both equations together, and $\overline{x+y}|^2=144$, *the square root of which* $x+y=12$; *substitute this for* $x+y$ *in the first, and* $\frac{12x}{y}=60$, *whence* $12x=60y$, *and* $x=5y$; *this value substituted for* x *in the equation* $x+y=12$, *gives* $6y=12$, *or* $y=2$; *whence* $x=(5y=)\ 10$.

10. Given $x\times\overline{x+y+z}=36$, $y\times\overline{x+y+z}=27$, and $z\times\overline{x+y+z}=18$, to find x, y, and z.

Add the three equations together, and the sum is $(\overline{x+y+z}\times\overline{x+y+z}=)\ \overline{x+y+z}|^2=81$; *extract the square root of this, and* $x+y+z=9$; *substitute this value in the three given equations, and* $9x=36$, *or* $x=4$; $9y=27$, *or* $y=3$; *and* $9z=18$, *or* $z=2$.

11. Given $\overline{x+y}\times3=540$, and $\frac{x-y}{4}=5$, to find x and y. *Ans.* $x=100$, $y=80$.

12. Given $x+y:x-y::8:5$, and $x+y:2y::8:3$, to find x and y. *Ans.* $x=65$, $y=15$.

13. Given $4x+\frac{y+z}{2}=54$; $3x+\frac{y+z}{3}=40$, and $2x-\frac{y+z}{4}=21$, to find x, y, and z. *Ans.* $x=12$, $y=9$, $z=3$.

14. Given $2\times\overline{x+y+z}=18$; $2\times\overline{x+y}+z=16$, and $2x+y+z=13$, to find x, y, and z. *Ans.* $x=4$, $y=3$, $z=2$.

15. Given $100-x-y=68$; $68-y-z=48$, and $48-x-z=20$, to find x, y, and z. *Ans.* $x=20$, $y=12$, $z=8$.

16. Given $\frac{2x+3y+4z}{5}=50$, $\frac{3x+4y+5z}{6}=56\frac{2}{3}$, and $\frac{5x+6y+7z}{8}=65$, to find x, y, and z. *Ans.* $x=40$, $y=30$, $z=20$.

REDUCTION OF AFFECTED QUADRATIC EQUATIONS.

96. An affected quadratic [k] equation is that which contains both the first and second powers of the unknown quantity; every equation of this kind is comprehended under one of the three following forms, viz.

$$\left.\begin{array}{ll}\text{First form,} & x^2+ax=b.\\ \text{Second form,} & x^2-ax=b.\\ \text{Third form,} & x^2-ax=-b.\end{array}\right\}\begin{array}{l}\text{Where } x \text{ is the unknown}\\ \text{quantity to be found, and } a\\ \text{and } b \text{ known quantities.}\end{array}$$

The value of the unknown quantity x, in each of these three forms, is found by one general method, as follows.

97. *To find the roots of affected quadratic equations.*

RULE I. Range the terms of the given equation according to the dimensions of the unknown quantity; namely, let the term containing the square stand in the *first place*, (to the left,) and that containing the first power in the *second*, on that side of the equation in which the square will be affirmative.

II. Transpose all the known quantities to the other side of the equation; and if the square of the unknown quantity have a multiplier or a divisor, it must be taken away by the methods employed in simple equations.

III. Take half the coefficient of the unknown quantity in the second term, square it, and add this square to both sides of the equation, then will that side which contains the unknown quantity be a complete square [l].

---

[k] *Quadratic* is derived from the Latin *quadratus*, squared. The term *adfected*, or *affected*, (from *affecto* to pester or trouble,) was introduced by Vieta, the great improver of Algebra, about the year 1600: it is used to distinguish equations which involve, or are *affected* with different powers of the unknown quantity, from those which contain one power only, which are therefore called *pure*. Dr. Hutton sometimes calls the former *compound* equations: this term the venerable and learned Baron Maseres highly approves of, observing that it is less obscure, and therefore more proper, than that of *affected* or *adfected* equations.

[l] Since the square of every simple quantity is a *simple* quantity, and the square of every binomial is a *trinomial*, it follows that no quantity in the form of a *binomial* can be a complete square; but that, in order to make it such, another term must be added to it, which term may in every case be found, from

IV. Extract the square root from both sides of the equation, prefixing to that of the known side the double sign $\pm$ [m].

V. Transpose the known part of the root; this incorporated (according to the import of its sign) with the root of the known side, will be the value required.

Note. The root of the unknown side is readily found by taking the roots of the first and third terms, and connecting them by the sign of the second [n].

---

the two given ones: to make this appear, let $a+b$ be any binomial, the square of which is $a^2+2ab+b^2$; now suppose a^2+2ab to be given, if half the coefficient $(2b)$ of a in the second term, viz. b, be taken, and squared, this square, viz. b^2, will be the remaining term; in like manner if b^2+2ab be given, if half the coefficient $(2a)$ of b in the second term be squared, its square a^2 will be the remaining term; wherefore, since this quantity being added to both sides of the equation does not affect their equality, this part of the rule is manifest.

[m] The reason for the double sign is this—every square has both an affirmative and a negative root; thus a^2 may arise either from the multiplication of $+a$ into $+a$, or $-a$ into $-a$; therefore a^2 has both $+a$ and $-a$, or $\pm a$ for its roots, and the same is evidently true in general.

[n] Because the square root of $a^2+2ab+b^2$ is $a+b$, and that of $a^2-2ab+b^2$ is $a-b$, it follows, that in every complete square, the root of the first term (a^2) connected with the root of the third term (b^2) by the sign of the middle term $(2ab)$, will be the root of the square: thus in the above example, $a+b$ is the root in the first instance, and $a-b$ is the root in the second.

In the first form of quadratics, where $x^2+ax=b$, x will always have two values, one affirmative, and one negative, and the negative value will be the greatest; for from the solution of the above equation, we have $x=\pm\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$; now the former of these values, namely, $+\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$, will be affirmative, since $\sqrt{b+\frac{1}{4}aa}$ is always greater than $\frac{1}{4}aa$, or its equal $\frac{1}{2}a$.

The second value, namely, $-\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$, being composed of two negative terms, will evidently be negative: moreover, since the affirmative root is the *difference* of the two quantities, ($\sqrt{b+\frac{1}{4}aa}$ and $\frac{1}{2}a$,) and the negative root their sum, it follows that the negative root will be the greatest.

In the second form, where $x^2-ax=b$, x will likewise have two values, one affirmative, and one negative, and the affirmative value will be the greatest; for from the solution $x=\pm\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, the first value of x, namely, $+\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, being composed of two affirmative terms, will evidently be affirmative. The second value of x, namely, $-\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, will always be negative, for since $b+\frac{1}{4}aa$ is greater than ($\frac{1}{4}aa$, or than its equal) $\frac{1}{2}a$, it follows that $-\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$ will be negative; and because

EXAMPLES.

1. Given $x^2+6x=40$, to find the values of x.

Half the coefficient of x in the second term, is $(\frac{6}{2}=)$ 3, *this squared is* $(3^2=)$ 9, *which being added to both sides, the equation becomes* $x^2+6x+9=(40+9=)$ 49.

The square root of which is $x+3=(\pm\sqrt{49}=)\pm 7$.

---

the former root is the *sum*, and the latter the *difference*, of two affirmative quantities, it follows that the affirmative root is the greatest.

In the first and second forms, the quantity under the radical sign is always affirmative, and therefore it can never happen that either of the roots in these two forms is impossible.

In the third form, where $x^2-ax=-b$, or $x=\pm\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, there are three cases; b may be either greater than $\frac{1}{4}aa$, equal to $\frac{1}{4}aa$, or less than $\frac{1}{4}aa$; if b be greater than $\frac{1}{4}aa$, the quantity under the radical sign will be negative; and since the square root of a negative quantity is impossible, both values of x will evidently in this case be impossible.

If b be equal to $\frac{1}{4}aa$, the quantity under the radical sign will be = nothing, and x will have but one value, namely, $\frac{1}{2}a$. But if b be less than $\frac{1}{4}aa$, x will have two values *both* affirmative; for the first value of x, namely, $+\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, will be the sum of two affirmative terms, and will therefore be affirmative.

The second value of x, namely, $-\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, will likewise be affirmative; for since $\frac{1}{4}aa$ is greater than b, it is plain that $\sqrt{\frac{1}{4}aa}$, or its equal $\frac{1}{2}a$, is greater than $\sqrt{\frac{1}{4}aa-b}$, and therefore $-\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$ will always be affirmative: therefore, when $x^2-ax=-b$, if $\frac{1}{2}a$ be greater than b, we shall have $x=+\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, and $x=-\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, both affirmative values of x: hence this is sometimes called the *ambiguous* form.

Either of these roots, whether affirmative, negative, or impossible, will answer the *algebraic* conditions of the equation from whence it is derived; but in the application of quadratics to the solution of problems, impossible roots always imply inconsistency in the conditions, or that the problem, as to any real use, is impossible.

The affirmative roots, in the first and second forms, are in most cases the answers to the question proposed; sometimes however the negative roots are to be taken, when the quadratic forms a subordinate part of some more extensive solution: in the application of algebra to geometry, both the affirmative and negative roots have place, each having a distinct and necessary import. Of the two affirmative roots in the third form, one only will for the most part be the answer to a numerical problem, the conditions of which will always point out which it is: in geometrical problems both roots have place, as we have already observed.

Whence, by transposition $x=(\pm 7-3=)\ 4$, or -10, *the answer.*

2. Given $x^2-8x=20$, to find x.

Half the coefficient 8 *is* 4, *this squared is* 16, *and the square added to both sides, gives* $x^2-8x+16=(20+16=)\ 36$.

By evolution $x-4=(\pm\sqrt{36}=)\ \pm 6$.

By transposition $x=(\pm 6+4=)\ 10$, *or* -2, *the answer.*

3. Given $x^2-4x=-3$, to find x.

Half the coefficient 4 *is* 2, *this squared is* 4, *which added to both sides, we have* $x^2-4x+4=(-3+4=)\ 1$.

Whence by evolution $x-2=(\pm\sqrt{1}=)\ \pm 1$.

And by transposition $x=(2\pm 1=)\ 3$, *or* 1, *the answer.*

4. Given $2x^2+16x=40$, to find x.

Divide by 2, *and* $x^2+8x=20$.

Complete the square, and $x^2+8x+16=(20+16=)\ 36$.

Extract the root, and $x+4=(\pm\sqrt{36}=)\ \pm 6$.

Whence, by transposition $x=(\pm 6-4=)\ 2$, *or* -10, *the ans.*

5. Given $5x^2-60x-12=788$, to find x.

By transposition $5x^2-60x=(788+12=)\ 800$.

By division $x^2-12x=160$.

Complete the square, and $x^2-12x+36=(160+36=)\ 196$.

Extract the root, and $x-6=(\pm\sqrt{196}=)\ \pm 14$.

By transposition $x=(6\pm 14=)\ 20$, *or* -8, *the answer.*

6. Given $8x^2+8x-4=12$, to find x.

By transposition $8x^2+8x=(12+4=)\ 16$.

By division $x^2+x=2$: *here the coefficient of* x *is* 1, *half of which is* $\frac{1}{2}$, *the square of which* $(\overline{\frac{1}{2}}|^2)$ *is* $\frac{1}{4}$; *therefore, to complete the square,* $x^2+x+\frac{1}{4}=(2\frac{1}{4}=)\frac{9}{4}$.

By evolution $x+\frac{1}{2}=(\pm\sqrt{\frac{9}{4}}=)\ \pm\frac{3}{2}$.

By transposition $x=(\pm\frac{3}{2}-\frac{1}{2}=\frac{\pm 3-1}{2}=)\ 1$, *or* -2, *the answer.*

7. Given $\frac{x^2}{3}-\frac{x}{4}+\frac{1}{5}=\frac{1}{6}$, to find x.

By transposition $\frac{x^2}{3}-\frac{x}{4}=(\frac{1}{6}-\frac{1}{5}=\frac{5-6}{30}=)-\frac{1}{30}$.

By multiplication $x^2-\frac{3x}{4}=\left(-\frac{3}{30}=\right)-\frac{1}{10}$.

By comp. the sq. $x^2-\frac{3}{4}x+\frac{9}{64}=\left(-\frac{1}{10}+\frac{9}{64}=\right)\frac{13}{320}$.

By evolution $x-\frac{3}{8}=\pm\sqrt{\frac{13}{320}}$.

By transposition $x=\left(\frac{3}{8}\pm\sqrt{\frac{13}{320}}=.375\pm\sqrt{.040625}=.375\right.$ $\left.\pm .20155, \&c.=\right) .57655$, &c. *or* $.17344$, &c. *the answer.*

8. Given $\frac{.012x}{.34x-.5}=\frac{.6x+.7}{.89x}$, to find x.

By multiplication $.01068x^2=.204x^2-.062x-.35$.

By transposition $.19332x^2-.062x=.35$.

By division $\left(x^2-\frac{.062x}{.19332}=\frac{.35}{.19332}, \text{ or}\right)$ $x^2-.3207x=$ 1.81046.

Completing the square, $x^2-.3207x+\overline{\frac{.3207}{2}}\Big|^2=1.81046+\overline{\frac{.3207}{2}}\Big|^2$.

By evolution $x-\frac{.3207}{2}=\pm\sqrt{1.81046+\overline{\frac{.3207}{2}}\Big|^2}$.

By transposition $x=\frac{.3207}{2}\pm\sqrt{1.81046+\overline{\frac{.3207}{2}}\Big|^2}=.16035$ $\pm\sqrt{1.81046+\overline{.16035}|^2}=.16035\pm\sqrt{1.81046+.0257121225}$ $=.16035\pm\sqrt{1.8361721225}=.16035\pm1.35505=1.5154$, *or* -1.1947, *the answer.*

9. Given $ax^2-bx+c=d$, to find x.

By transposition $ax^2-bx=d-c$.

By division $x^2-\frac{b}{a}x=\frac{d-c}{a}$.

By comp. the sq. $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$.

By evolution $x-\frac{b}{2a}=\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$.

By transposition $x=(\frac{b}{2a}\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}=)$ $\frac{b\pm\sqrt{4ad-4ac+b^2}}{2a}$, *the answer.*

10. Given $5x-3y=x+y+20$, and $x^2+y^2=11x-5y$, to find x and y.

From the first equation $x=(\frac{4y+20}{4}=)$ $y+5$; *whence* $x^2=(\overline{y+5})^2=)$ $y^2+10y+25$: *substitute these values respectively for* x *and* x^2 *in the second equation, and it becomes* $2y^2+10y+25=6y+55$.

By transposition $2y^2+4y=30$.
By division $y^2+2y=15$.
By comp. the sq. $y^2+2y+1=16$.
By evolution $y+1=\pm4$.
By transposition $y=(\pm4-1=)$ 3, *or* -5.
Whence $x=(y+5=3+5$, *or* $-5+5=)$ 8, *or* 0.

11. Given $x^2+xy=35$, and $xy-2y^2=2$, to find x and y.

Let $vy=x$: *this value substituted for* x *in both equations, they become* $v^2y^2+vy^2=35$, *whence* $y^2=\frac{35}{v^2+v}$; *and* $vy^2-2y^2=2$, *whence* $y^2=\frac{2}{v-2}$; *therefore* $\frac{35}{v^2+v}=\frac{2}{v-2}$; *whence*

By multiplication $35v-70=2v^2+2v$.
By transposition $2v^2-33v=-70$.
By division $v^2-16.5v=-35$.
By comp. the sq. $v^2-16.5v+68.0625=33.0625$.
By evolution $v-8.25=(\pm\sqrt{33.0625}=)$ ±5.75.
By transposition $v=(8.25\pm5.75=)$ 14, *or* 2.5.

Whence by taking $v=14$, *we have* $y=(\sqrt{\frac{2}{v-2}}=)\sqrt{\frac{1}{6}}$; *and* $x=(vy=)$ $14\sqrt{\frac{1}{6}}$; *and by taking* $v=2.5$, *we shall have* $y=(\sqrt{\frac{2}{v-2}}=\sqrt{\frac{2}{.5}}=)$ $\sqrt{4}=2$, *and* $x=(vy=)$ $2.5\times2=5$.

12. Given $x+y+xy=19$, and $x^2y+xy^2=84$, to find x and y.

Let $s=x+y$, *and* $p=xy$, *then the given equations will become* $(x+y+xy=)$ $s+p=19$; *and* $x^2y+xy^2=(\overline{x+y}\times xy=)$ $sp=84$: *from the square of the equation* $s+p=19$, *take four times*

*$sp=84$, and you will have $s^2-2sp+p^2=25$; whence by evolution $s-p=\pm 5$: this equation added to, and subtracted from $s+p=19$, gives $2s=(19\pm 5=)$ 24 or 14, and $s=12$ or 7; and likewise $2p=(19\mp 5=)$ 14 or 24, and $p=7$ or 12; therefore $x+y=(s=)$ 12 or 7, and $xy=(p=)$ 7 or 12. Subtract four times the last from the square of the last but one, and there remains $x^2-2xy+y^2=(\overline{12}|^2-28$, or $\overline{7}|^2-48=)$ 116 or 1; whence, by evolution, $x-y=\pm\sqrt{116}$, or ± 1; add this to, and subtract it from $x+y=12$ or 7, and we shall have $2x=\pm\sqrt{116}+12$, or $\pm 1+7=$ (by taking the latter value only) 8 or 6, whence $x=4$ or 3; likewise $2y=(7\mp 1=)$ 6 or 8, whence $y=3$ or 4; if we make $x=4$, then $y=3$; if $x=3$, then $y=4$ **.

13. Given $\frac{x^2}{y}+\frac{y^2}{x}=9$, and $x+y=6$, to find x and y.

Let $x=z+v$, and $y=z-v$, then by adding these two equations together, $(x+y=)$ $2z=6$, and $z=3$; whence $x=3+v$, and $y=3-v$. Multiply the first given equation by xy, and $x^3+y^3=9xy$; which by substituting $3+v$ for x, and $3-v$ for y, becomes $\overline{3+v}|^3+\overline{3-v}|^3=9\times\overline{3+v}\times\overline{3-v}$; this by involution, multiplication, and addition, becomes $54+18v^2=81-9v^2$, whence by transposition $27v^2=27$; therefore $v^2=1$, and $v=\pm 1$, whence $x=(z+v=3\pm 1=)$ 4 or 2; and $y=(z-v=3\mp 1=)$ 2 or 4; if $x=4$, then $y=2$; but if $x=2$, then $y=4$.

14. Given $x^2+8x=65$, to find x. *Ans.* $x=5$, *or* -13.

15. Given $y^2-12y=540$, required the value of y? *Ans.* $y=30$, *or* -18.

16. Given $z^2-20z=-91$, what is z equal to? *Ans.* $z=13$, *or* 7.

17. Given $3x^2-21x-450=6000$, to find x. *Ans.* $x=50$, *or* -43.

18. Given $z^2+z=2$, required the value of z? *Ans.* $z=1$, *or* -2.

---

* If the former values of $x-y$, namely, $\pm\sqrt{116}$ or ± 10.77 be taken, then for the affirmative values $x=11.385$, &c. and $y=.615$, &c. and for the negative values $x=.615$, &c. and $y=11.385$, &c. both of which values answer the conditions of the question equally with those given above. It appears from the solution, that this example (which was inserted by mistake) is not an *affected*, but a *pure* quadratic, and therefore is misplaced; the same may be said of the thirteenth example.

19. Given $5y^2-25y+40=10$, to find y. *Ans.* $y=3$, *or* 2.

20. Given $x^2-\frac{5}{6}x+\frac{1}{6}=0$, to find x. *Ans.* $x=\frac{1}{2}$ *or* $\frac{1}{3}$.

21. Given $\frac{x^2}{4}-\frac{x}{5}-\frac{1}{6}=\frac{1}{7}$, to find x. *Ans.* $x=1.5824$, *or* $-.7824$.

22. Given $ax^2+bx=c$, to find x. *Ans.* $x=-\frac{b}{2a}\pm\frac{1}{2}a\sqrt{4ac+b^2}$.

23. Given $\frac{2x^2-3}{4}+\frac{5x+6}{7}=\frac{8x+9}{11}$, to find x. *Ans.* $x=1.205565$, *or* -1.179593.

24. Given $2x^2+x-y=3y^2+2y$, and $x+y=7$, to find x and y. *Ans.* $x=42$ *or* 4, $y=-35$ *or* 3.

98. By this rule may be solved all equations whatever, wherein there are only two different dimensions of the unknown quantity, provided the index of the one be exactly double that of the other.

RULE. Having completed the square and extracted the root as before, transpose the known quantities which are on the same side with the unknown one, and then extract the root implied by the index of the unknown quantity, from both sides of the equation [p].

25. Given $x^4+6x^2=72$, to find the values of x.

By completing the square $x^4+6x^2+9=81$.

By evolution $x^2+3=\pm 9$.

By transposition $x^2=(\pm 9-3=)\,6$ *or* -12.

By evolution $x=(\pm\sqrt{6}=)\pm 2.4494897428$, *or* $\pm\sqrt{-12}$, *the latter of which are impossible.*

---

[p] Every equation will have as many roots as the unknown quantity has dimensions; thus, in the 25th example, x being in the fourth power, the equation will have four roots, as appears by the solution. In example 26, y comes out equal to 49 or 25, the latter of which being substituted in the equation, will not answer, except -5 be substituted for the square root instead of $+5$, the reason of which is obvious, since the 25 arose from -5×-5. One root only is required in the following examples, as finding the rest would, in many cases, require rules which have not yet been given.

26. Given $y-4\sqrt{y}=45$, to find y.
By comp. the sq. $y-4\sqrt{y}+4=49$.
By evolution $\sqrt{y}-2=\pm 7$.
By transposition $\sqrt{y}=(\pm 7+2=)\ 9$, *or* -5.
By involution $y=(\overline{9}|^2=)\ 81$, *or* $(-\overline{5}|^2=)\ 25$.

27. Given $ax^n-bx^{\frac{n}{2}}-c=-d$, to find x.

By trans. and division $x^n-\frac{b}{a}x^{\frac{n}{2}}=\frac{c-d}{a}$.

By comp. the square $x^n-\frac{b}{a}x^{\frac{n}{2}}+\frac{b^2}{4a^2}=(\frac{c-d}{a}+\frac{b^2}{4a^2}=)\ \frac{4ac-4ad+b^2}{4a^2}$.

By evolution $x^{\frac{n}{2}}-\frac{b}{2a}=\pm\sqrt{\frac{4ac-4ad+b^2}{4a^2}}$.

By transposition $x^{\frac{n}{2}}=(\frac{b}{2a}\pm\sqrt{\frac{4ac-4ad+b^2}{4a^2}}=)\ \frac{b\pm\sqrt{4ac-4ad+b^2}}{2a}$.

By evolution $x=\left.\frac{b\pm\sqrt{4ac-4ad+b^2}}{2a}\right|^{\frac{2}{n}}$.

28. Given $x^4+2x^2=24$, to find one value of x. *Ans.* $x=2$.
29. Given $y^6-4y^3=32$, to find y. *Ans.* $y=2$.
30. Given $z-2\sqrt{z}=0$, to find z. *Ans.* $z=4$.
31. Given $2x^4-x^2=496$, to find x. *Ans.* $x=4$.
32. Given $x^{2n}-x^n=a$, to find x. *Ans.* $x=\sqrt[n]{\frac{1\pm\sqrt{4a+1}}{2}}$.

99. PROBLEMS[q].

Every problem proposed to be solved algebraically, contains some conditions laid down, which are called the *data*; from whence one or more quantities are required to be found, called the *quæsita*. The first thing necessary to be done preparatory to the solution, is to understand clearly the import and signification of the problem: it must be freed from every thing ambiguous and un-

---

[q] The word *problem* is derived from the Greek πρόβλημα. An algebraic problem is a proposition wherein some unknown truth is required to be investigated or discovered, and the truth of the discovery demonstrated.

necessary; the conditions, and the manner of their dependance on each other, must be clearly ascertained and stated, and they must be carefully distinguished from the quantities proposed to be found: when this is accomplished, the conditions of the proposed problem will be exhibited under the form of one or more equations; namely, as many equations as there are unknown quantities, the solution of which is the subject of the preceding rules.

Much depends on a proper substitution for the quantities required: no general rule for this can be given; sometimes a letter must be put for each; frequently, having substituted a letter for one of the unknown quantities, expressions for the others may be derived by means of this and the conditions proposed, without the aid of new letters; sometimes a substitution for the sum, difference, product, quotient, roots, powers, &c. of the unknown quantities, may be conveniently made; but the proper application of these must be learned by experience and practice.

The following modes of substitution will apply in many cases. For one unknown quantity put x, for two put x and y, x being the greater, y the less; for their sum $x+y$, for their difference $x-y$, for the square of the greater x^2, for the cube root of the less $\sqrt[3]{y}$, for the sum of their squares x^2+y^2, for the difference of their squares x^2-y^2, for the square of the sum $\overline{x+y}|^2$, for the cube root of their difference $\sqrt[3]{x-y}$, for their product xy, their quotient $\frac{x}{y}$, where the greater is proposed to be divided by the less, or $\frac{y}{x}$, where the less is proposed to be divided by the greater.

In general, the sum of any two quantities is represented by interposing the sign $+$ between them; the difference by the sign $-$, the product by the sign $\times$, or by placing them as coefficients to each other, and the quotient by placing the dividend above the divisor; following in every case the method applicable to it, as proposed in algebraic notation.

1. What number is that, to which 9 being added, the sum will be 23?

Let the required number be represented by x.

To which adding 9, the sum will be $x+9$.

This, by the problem, equals 23, whence $x+9=23$.

Wherefore by transposition $x = (23-9=)\ 14$, *the answer required* [r].

2. What number is that, from which 27 being subtracted, the remainder is 41?

Let the number required be called x.

From which subtracting 27, *the remainder will be* $x-27$.

This, by the problem, equals 41, *whence* $x-27=41$.

Therefore, by transposition $x = (41+27=)\ 68$, *the answer required* [s].

3. What number is that, which being multiplied by 4, and 5 being subtracted from the product, the remainder will be 6?

Let the required number be x.

This multiplied by 4, *is* $4x$.

From which subtracting 5, *the remainder is* $4x-5$.

This remainder by the problem equals 6, *wherefore* $4x-5=6$.

By transposition $4x = (6+5=)\ 11$.

And by division $x = (\frac{11}{4}=)\ 2\frac{3}{4}$, *the answer* [t].

4. What number is that, which being divided by 7, with 8 added to the quotient, the sum will be 9?

Let the required number be called x.

This divided by 7, *the quotient is* $\frac{x}{7}$.

To this quotient adding 8, *it becomes* $\frac{x}{7}+8$.

Which sum, by the problem, equals 9, *whence* $\frac{x}{7}+8=9$.

---

[r] *Proof of prob.* 1. The number required is 14
For by adding to it 9
The sum is .... 23 as required.

[s] *Proof of prob.* 2. The number required is 68
For by subtracting from it 27
The remainder is .... 41 as was required.

[t] *Proof of prob.* 3. The required number is $2\frac{3}{4}$, that is $\frac{11}{4}$.
For if it be multiplied by 4, the product is 11.
From this subtracting 5, the remainder is $11-5=6$, according to the problem.

Whence by transposition $\frac{x}{7} = (9-8=)\ 1$.

And by multiplication $x=7$, *the answer* [w].

5. What number is that, from which 12 being subtracted, one fourth of the remainder will be 22?

Let the number required be represented by x.

From which subtracting 12, *the remainder is* $x-12$.

One fourth of this remainder is $\frac{x-12}{4}$.

This by the problem equals 22, *whence* $\frac{x-12}{4}=22$.

Wherefore by multiplication $x-12=88$.

And by transposition $x=(88+12=)\ 100$, *the answer* [x].

6. What number is that, to which 7 being added, two thirds of the sum will be 8?

Let the number be called x.

To which adding 7, *the sum is* $x+7$.

Two thirds of this is $(\frac{2}{3}\times\overline{x+7}=)\ \frac{2x+14}{3}$.

This equals 8 *by the problem, whence* $\frac{2x+14}{3}=8$.

By multiplication $2x+14=24$.

By transposition $2x=(24-14=)\ 10$.

And by division $x=5$, *the answer* [y].

7. What number is that, which being added to 16, and subtracted from 20, the remainder will be two sevenths of the sum?

---

[w] *Proof.* The number required is 7.
For this, divided by 7, viz. $\frac{7}{7}$, is 1.
To which adding 8, the sum is $1+8=9$, according to the problem.

[x] *Proof.* The number required is 100.
From which subtracting 12, it becomes $100-12=88$.
One fourth of this remainder, or $\frac{88}{4}$ is 22, as was proposed.

[y] *Proof.* The number required is 5.
To which adding 7, the sum is $5+7$ or 12.
Two thirds of this sum is $\frac{2}{3}\times 12=(\frac{2\times 12}{3}=)\frac{24}{3}=8$, as was proposed.

Let the number be called x.
This added to 16, the sum is $x+16$.
Two-sevenths of which is $(\frac{2}{7}\times\overline{x+16}=)\ \frac{2x+32}{7}$.
Also subtracting x from 20, the remainder is $20-x$.
Wherefore by the problem $20-x=\frac{2x+32}{7}$.
This by multiplication becomes $140-7x=2x+32$.
And by transposition $9x=108$.
Whence by division $x=12$, *the answer* [a].

8. What number is that, of which its one-fourth part exceeds its one-fifth part by 4.
Let the number sought be x.
Then its one-fourth part will be $\frac{x}{4}$.
And its one-fifth part $\frac{x}{5}$.
Whence by the problem $\frac{x}{4}=\frac{x}{5}+4$.
By multiplication $5x=4x+80$.
Whence by transposition $x=80$, *the answer* [a].

9. A lady being asked her age, replied, 'If you add $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of my age together, the sum will be 18:' how old was she?
Let the lady's age be called x.
Then will its $\frac{1}{3}$ *be* $\frac{x}{3}$, *its* $\frac{1}{4}=\frac{x}{4}$, *and its* $\frac{1}{6}=\frac{x}{6}$.
Therefore by the problem $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=18$.

---

[a] *Proof.* The required number is 12; for by adding 16 to it, the sum is $12+16=28$; also subtracting the given number 12 from 20, the remainder is $20-12=8$: now 8 is two-sevenths of 28, for $28\times\frac{2}{7}=(\frac{28\times 2}{7}=)\ \frac{56}{7}=8$, as was proposed.

[a] *Proof.* The number to be found is 80; for one-fourth part of it is $\frac{80}{4}=20$, and one-fifth part $\frac{80}{5}=16$: now 20 exceeds 16 by 4, which was to be shewn.

Whence by multiplication by 12 $(4x+3x+2x=)\ 9x=216$.

And by division $x=24$ *years, the answer* [b].

10. Divide 17 shillings between two persons, so that one may have 4 shillings more than the other.

Let the less share be called x.

Then will the greater be $x+4$.

And both shares added together, will be $2x+4$.

This by the problem equals 17, *or* $2x+4=17$.

Whence by transposition $2x=(17-4=)\ 13$.

And by division $x=(\frac{13}{2}=6\frac{1}{2}=)$ 6*s*. 6*d*. = *the least share; subtract this from* 17, *and* (17—6*s*. 6*d*.=) 10*s*. 6*d*.= *the greater share* [c].

Or thus,

Let the greater share be x.

Then the less will be $x-4$.

The sum of these equals 17, *viz.* $2x-4=17$.

Whence $2x=21$, $x=$ 10*s*. 6*d*. *and* $x-4=$ 6*s*. 6*d*. *as before.*

11. Three persons, A, B, and C, rent 140 acres of land between them, of which A has twice as much as B, and B thrice as much as C; how many acres has each?

Let C's number be called x.

Then B's will be (=*thrice C's, or*) $3x$.

And A's (=*twice B's, or*) $6x$.

And the sum of these, by the problem, $10x=140$.

---

[b] *Proof.* The lady's age is 24:

For $\frac{1}{3}$ of it, or $\frac{24}{3}=8$.

$\frac{1}{4}$ of it, or $\frac{24}{4}=6$.

$\frac{1}{6}$ of it, or $\frac{24}{6}=4$

Their sum $=18$, as required.

[c] *Proof.*

	s.	*d.*
The share of one is	10	6
That of the other ..	6	6
The sum of both is	17	0
And the difference	4	0

agreeably to the conditions proposed.

Whence $x=14=C$*'s share*, $3x=(3\times14=)\ 42=B$*'s share, and* $6x=(6\times14=)\ 84=A$*'s share* [d].

12. If the half, third, and fourth parts of my number of shillings be added together, the sum will be one shilling more than I have; how many shillings have I got?

Let x *= the number required, then will* $\frac{x}{2}$ *= the half,* $\frac{x}{3}$ *= the third part, and* $\frac{x}{4}$ *= the fourth part; also* $x+1$ *= one more than I have; whence by the problem* $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=x+1$; *this equation cleared of fractions, becomes* $(6x+4x+3x=)\ 13x=12x+12$; *wherefore* $x=12$, *the number required* [e].

13. A legacy of 120*l.* was left between A and B, in such sort, that $\frac{1}{8}$ of A's share was equal to $\frac{1}{7}$ of B's; what sum did each receive?

Let $8x=A$*'s share, then will* $7x=B$*'s, and their sum* $=(8x+7x=)\ 15x=120$ *by the problem; whence* $x=(\frac{120}{15}=)\ 8$: *consequently* $8x=64l.=A$*'s share, and* $7x=56l.=B$*'s.*

14. A post is $\frac{1}{4}$ in the mud, $\frac{1}{5}$ in the water, and 11 feet above the water; required the length of the post?

Let its length be called x, *then the* $\frac{1}{4}$ *part will be* $\frac{x}{4}$, *and the* $\frac{1}{5}$ *part* $\frac{x}{5}$; *whence by the problem* $\frac{x}{4}+\frac{x}{5}+11=x$: *this equation cleared of fractions, &c. we shall have* $x=20$ *feet;* $\frac{x}{4}=5$ *feet in the mud, and* $\frac{x}{5}=4$ *feet in the water.*

---

[d] *Proof.* A's, B's, and C's shares added together, viz. $84+42+14=140$, which is one condition of the problem.

Also A's share is double of B's, for $84=2\times42$; and B's share is triple of C's, for $42=3\times14$, which is the other condition.

[e] *Proof.* The number obtained by the solution is 12; the half of which is 6, the third 4, and the fourth 3: now $6+4+3=13=12+1$; that is, the sum of the half, third, and fourth, exceeds the number I have by 1, which was to be shewn. In like manner the truth of the conclusions in all the following problems may be demonstrated.

15. A farmer turned 152 beasts into a meadow; to every horse there were 3 cows, and to every cow, 5 sheep; how many of each sort were there?

Suppose there were x horses, then would there be $3x$ cows, and $(3x\times 5=)$ $15x$ *sheep; whence by the problem* $(x+3x+15x=)$ $19x=152$: *therefore* $x=8$ *horses,* $3x=24$ *cows, and* $15x=120$ *sheep.*

16. A man has six sons, whose successive ages differ by 4 years, and the eldest is thrice as old as the youngest; required their several ages?

Let the age of the youngest be x.
Then will that of the second youngest be $x+4$.
that of the third .... $x+8$.
that of the fourth .... $x+12$.
that of the fifth .... $x+16$.
that of the sixth .... $x+20$.

Whence by the problem, $x+20=3x$, *or* $2x=20$, *and* $x=10$ *years* $=$ *the age of the youngest; also* $x+4=(10+4=)\,14=$ *the age of the second,* $x+8=18=$ *that of the third,* $x+12=22=$ *that of the fourth,* $x+16=26=$ *that of the fifth, and* $x+20=30=$ *the age of the eldest.*

17. Two butchers bought a calf for 40 shillings, of which the part paid by A, was to the part paid by B, as 3 to 5; what sum did each pay?

Let $x=A$*'s part, then* $3:5::x:\frac{5x}{3}=B$*'s part; whence by the problem* $(x+\frac{5x}{3}=\frac{3x+5x}{3}=)\,\frac{8x}{3}=40$; *therefore* $8x=120$, *and* $x=15$ *shillings paid by A; also* $\frac{5x}{3}\,(=\frac{5\times 15}{3}=\frac{75}{3}=)\,25$ *shillings paid by B.*

Or thus,

Let $3x=A$*'s part, then will* $5x=B$*'s, and their sum* $8x=40$, *whence* $x=5$; *therefore* $3x=15s.=$ *sum paid by A, and* $5x=25s.=$ *sum paid by B, the very same as before.*

18. A person rented a house on a lease of 21 years, and agreed to do the repairs when $\frac{2}{3}$ of the part of the lease elaps-

ed, should equal $\frac{8}{9}$ of the part to come; how long will he have been in possession when the repairs are begun?

Let x=the time elapsed, then will $21-x$=the time to come; also $\frac{2x}{3}$=two thirds of the time elapsed; and ($\frac{8}{9}\times\overline{21-x}=$) $\frac{168-8x}{9}$=eight ninths of the time to come; whence by the problem, $\frac{2x}{3}=\frac{168-8x}{9}$; this cleared of fractions, is $18x=504-24x$, whence $42x=504$, and $x=12$ years, the time required.

19. Two country girls went to a fair, A laid out as much above 4 shillings, as B did under 6; and the sum spent by A was to that spent by B as 7 to 8; how much did each lay out?

Let $4+x$ be the sum A spent, then will $6-x$ be the sum B spent; then by the problem $4+x:6-x::7:8$: whence by multiplying extremes and means, ($\overline{4+x}\times8=\overline{6-x}\times7$, or) $32+8x=42-7x$; therefore $15x=10$, and $x=$ ($\frac{10}{15}=\frac{2}{3}$ of a shilling$=$) 8 pence; whence $4+x=$4s. 8d.=sum spent by A, and $6-x=$ (6s.—8d.=) 5s. 4d.=sum spent by B.

20. The sum of the ages of a man and his wife is 55 years, and his age exceeds her's by 7 years; required the age of each?

Let x=the man's age, then $55-x$=the woman's; subtract the latter from the former, and $2x-55=7$ by the problem; whence $2x=62$, and $x=31$ years=the man's age, therefore $55-x=(55-31=)$ 24 years=the wife's age.

Or thus,

Let x=the wife's age, then $55-x$=the man's; subtract the former from the latter, and $55-2x=7$ by the problem; whence $2x=(55-7=)$ 48, and $x=24$=the woman's age; also $55-x=(55-24=)$ 31=the man's age, as before.

Or thus,

Let x=the man's age, then $x-7$=the woman's; add both equations together, and $2x-7=55$ by the problem; whence $x=31$, and $x-7=24$, as before.

Or thus,

Let x=the woman's age, then $x+7$=the man's; add both equations together, and $2x+7=55$ by the problem; whence $x=24$, and $x+7=31$, as before.

Or thus,

Let x = *the man's age,* y = *the woman's; then by the problem* $x+y=55$, *and* $x-y=7$; *this may be solved by each of the three methods in Art.* 90, 91, *and* 92, *whence* x *will be found* $=31$, *and* $y=24$, *as before.*

21. What two numbers are those, the sum of which is 140; and if four times the less be subtracted from three times the greater, the remainder is 70.

Let x = *the greater, then* $140-x$ = *the less;* 3 *times the greater* $=3x$, *and* 4 *times the less* $=(4\times\overline{140-x}=)\ 560-4x$; *therefore by the problem* $(3x-\overline{560-4x}=)\ 7x-560=70$; *whence* $7x=630$, *and* $x=90$ = *the greater, also* $140-x=(140-90=)$ 50 = *the less.*

22. A labourer received his week's wages, amounting to twenty shillings, in half crowns and sixpences, and there were twenty pieces in all; how many of each did he receive?

Let x = *the number of half crowns, then* $20-x$ = *the number of sixpences, also* $5x$ = *the number of sixpences in* x *half crowns, and* 40 = *the number in* 20 *shillings; therefore* $(5x+20-x=)$ $4x+20=40$ *by the problem; whence* $4x=20$, *and* $x=5$ = *the number of half crowns, and* $20-x=(20-5=)\ 15$ = *the number of sixpences.*

23. A paid B 20 guineas, and then B had twice as much money as A had left; but if B had paid A 20 guineas, A would have had thrice as much as B had left; what sum did each possess at first?

Let x = *A's sum at first,* y = *B's, then* $x-20$ = *A's remaining sum, and* $y+20$ = *B's sum after the payment was made; also* $x+20$ = *A's sum, and* $y-20$ *B's remainder, had B paid A* 20 *guineas; whence by the problem* $y+20=(2\times\overline{x-20}=)\ 2x-40$, *and* $x+20=(3\times\overline{y-20}=)\ 3y-60$; *from the former* $y=2x-60$, *and from the latter* $y=\frac{x+80}{3}$; *whence* $2x-60=\frac{x+80}{3}$, *or* $6x-180$ $=x+80$, *or* $5x=260$, *therefore* $x=52$ *guineas; also* $y=2x-60$ $=(104-60=)\ 44$ *guineas.*

24. It is twelve o'clock, and the hour and minute hands of my watch are exactly together; at what o'clock will they be next together, and how often does the minute hand pass the hour hand in twelve hours?

Let $x=$ the space the hour hand has passed (or its distance from 12) when they come together: now it is evident that the minute hand must go once round the dial besides the space x, in order to overtake the hour hand; whence $1+x=$ the space through which the minute hand moves in the same time. Likewise the minute hand moves 12 times as fast as the hour hand, wherefore $12:1::1+x:x$, by the problem; whence $12x=1+x$, or $11x=1$, or $x=\frac{1}{11}$ of 12 hours, $=\frac{12}{11}$ of 1 hour $=1\frac{1}{11}$ hour $=1h.\ 5m.\ \frac{5}{11}$ $=5\frac{5}{11}$ minutes past one o'clock; also $12\div\frac{12}{11}=12\times\frac{11}{12}=11$, the number of times the minute hand passes the other in 12 hours.

25. Two beggars went to an alehouse to share their booty; after dividing it equally, A spent 5*d.* and B 8*d.* they then tossed up, and A won 20*d.* of B, after which A's cash was double of B's; what sum had each at first?

Let $x=$ the share of each.

Then $\left.\begin{array}{l} x-5=A\text{'s sum} \\ x-8=B\text{'s sum}\end{array}\right\}$ *after spending.*

And $\left.\begin{array}{l} (x-5+20=)\ x+15=A\text{'s sum} \\ (x-8-20=)\ x-28=B\text{'s sum}\end{array}\right\}$ *after tossing up.*

Therefore $x+15=(2\times\overline{x-28}=)\ 2x-56$ by the problem; whence $x=71$ pence $=5s.\ 11d.=$ the share of each.

26. What fraction is that, which if 1 be added to its numerator, the value of the fraction will be $\frac{1}{3}$; but if 1 be added to its denominator, the value will be $\frac{1}{4}$?

Let $x=$ the numerator, $y=$ the denominator, then $\frac{x}{y}=$ the fraction; wherefore by the problem $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$.

From the first equation we have $x=\frac{y}{3}-1$, and from the second $x=\frac{y+1}{4}$, whence $\frac{y}{3}-1=\frac{y+1}{4}$; from this by reduction we get $y=15$, and consequently $x=(\frac{y+1}{4}=)\ 4$, whence $\frac{x}{y}=\frac{4}{15}=$ the fraction required.

27. A Gentleman left 56*l.* between two persons, whose shares were respectively as $\frac{2}{7}$ to $\frac{3}{8}$; what sum did each receive?

Let x *= the least share, then* $56-x$ *= the greater, and by the problem* $x : 56-x :: \frac{2}{7} : \frac{3}{8}$, *whence* $\frac{3x}{8}=\frac{112-2x}{7}$; *therefore* $x=(\frac{896}{37}=24l.2162=)$ $24l.\ 4s.\ 3d\frac{1}{4}$. *and* $56-x=(56-24.2162=31.7838=)$ $31l.\ 15s.\ 8d\frac{1}{4}$.

28. Some fishermen having caught a shark, and cut it into three pieces, the tail weighed 60 pounds; the head weighed as much as the tail and $\frac{1}{4}$ the body; and the body weighed as much as the head and tail together; required the weight of the shark, and of each of its parts?

Let x *= the weight of the body, then* $60+\frac{x}{4}$ *= weight of the head; and* $(60+\frac{x}{4}+60=)$ $\frac{x}{4}+120=x$, *by the problem; whence* $x=160$ *pounds = the weight of the body;* $60+\frac{x}{4}=(60+40=)$ 100 *pounds = the weight of the head; and* $(160+100+60=)$ 320 *pounds = the weight of the shark.*

29. A hare is 50 of her own leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's: how many leaps must each take before the hare is caught?

Let x *= the greyhound's leaps,* y *= the hare's after the dog's starting, then* $y+50$ *= the whole of the hare's leaps. But* $3 : 4 :: x : y$ *by the problem, whence* $3y=4x$, *and* $y=\frac{4x}{3}$, *also* $y+50=\frac{4x}{3}+50$; *whence by the problem* $2 : 3 :: x : \frac{4x}{3}+50$, *therefore* $\frac{8x}{3}+100=3x$; *whence* $x=300$ *= the greyhound's leaps, and* $\frac{4x}{3}+50=450$ *= the hare's leaps.*

30. A market-woman bought a number of eggs at 2 a penny, and as many at 3 a penny, and sold them all at 5 for twopence, whereby she lost fourpence; how many eggs had she in all?

Let x *= the number of each sort, then* $2x$ *= her whole stock, also* $\frac{x}{2}$ *= the value of those at 2 a penny, and* $\frac{x}{3}$ *= the value of those at 3 a penny; but* $5 : 2 :: 2x : \frac{4x}{5}$ *= the value of the whole*

at 5 for twopence; whence by the problem $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$, *consequently* $x=120=$ *the number of each sort.*

31. How must I divide the number 39 into four parts, so that the first part being increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the sum, difference, product, and quotient, may be equal to each other?

Let $x-1=$ *the first part, then* $x+2=$ *the second,* $\frac{x}{3}=$ *the third, and* $4x=$ *the fourth: then by the problem* $(x-1+x+2+\frac{x}{3}+4x=)\ 6x+\frac{x}{3}+1=39$; *this resolved gives* $x=6$, *whence* $x-1=5$, $x+2=8$, $\frac{x}{3}=2$, *and* $4x=24$, *the parts required.*

32. A sets out from London, and travels towards Carlisle, at the rate of $2\frac{1}{2}$ miles an hour; B sets out at the same time from Carlisle, and travels towards London, at the rate of $3\frac{1}{4}$ miles an hour; in how many hours, and whereabouts on the road will they meet, supposing the distance of the two places rom each other to be 301 miles?

Let $x=$ *the time required, then* $(2\frac{1}{2}\times x=)\ 2x+\frac{x}{2}=$ *A's distance travelled, and* $(3\frac{1}{4}\times x=)\ 3x+\frac{x}{4}=$ *B's distance, whence by the problem* $(2x+\frac{x}{2}+3x+\frac{x}{4}=)\ 5x+\frac{3x}{4}=301$; *this resolved gives* $x=52\frac{8}{23}$ *hours, and* $52\frac{8}{23}\times 2\frac{1}{2}=(\frac{1204}{23}\times\frac{5}{2}=)130\frac{20}{23}$ *miles from London, or* $(301-130\frac{20}{23}=)170\frac{3}{23}$ *miles from Carlisle.*

33. Farmer Trott and his Wife, ev'ry week of their life,
Us'd to drink out a firkin of Ale;
How comes it about, says Trott, when I'm out,
That it lasts eighteen days without fail?
But you're going to Gloster to visit Aunt Foster,
And then I've a fancy to see,
If I drink at the rate that I have done of late,
How long the same quantum serves me.

While the visit wife made, to the Aunt aforesaid,
House and farming Trott superintended,
If he drank of his beer, as he'd done all the year,
In what time would his firkin be ended?

Let x=the number of days it would last him, then x days : 1 firkin : : 7 days : $\frac{7}{x}$=the quantity drank by him in a week; and 18 days : 1 firkin : : 7 days : $\frac{7}{18}$=the quantity drank by her in a week; therefore $\frac{7}{x}+\frac{7}{18}=1$ by the problem, whence $x=11\frac{5}{11}$ days =the time it serves him.

34. In the ruins of the ancient city of Palmyra, there were found two cubical blocks of Granite, containing together 728 cubic feet, and the side of the less was to that of the greater, as 3 to 4; required the side of each?

Let x=the side of the less cube, then $(3:4::x:)\ \frac{4x}{3}$= the side of the greater; whence $(x^3+\overline{\frac{4x}{3}}\big|^3=)\frac{91x^3}{27}=728$; whence $91x^3=728\times27$, and $x^3=(\frac{728\times27}{91}=)\ 8\times27$, therefore $x=(\sqrt[3]{8\times27}=2\times3=)$ 6 feet=the side of the less cube, and $\frac{4x}{3}=(\frac{24}{3}=)$ 8 feet=the side of the greater.

35. A Jockey has two horses, A and B; he has also two saddles, one worth 16*l.* the other worth 4*l.* now if he puts the best saddle on A, and the worst on B, A will be worth twice as much as B: but if he puts the best saddle on B, and the worst on A, then B will be worth thrice as much as A; required the value of each horse?

Let x=the value of A, y=the value of B.

$$\left.\begin{aligned}&\textit{Then } x+16=(2\times\overline{y+4}=)\ 2y+8\\ &\textit{And } y+16=(3\times\overline{x+4}=)\ 3x+12\end{aligned}\right\}\ \textit{by the problem.}$$

From the first $x=2y-8$, and from the second $x=\frac{y+4}{3}$, con-

sequently $2y-8=\frac{y+4}{3}$, *whence* $y=(5\frac{3}{5}=)$ 5*l*. 12*s*.=*the price of B, and* $x=(3\frac{1}{5}=)$ 3*l*. 4*s*.=*the price of A*.

36. A privateer having taken a prize, its value was divided equally among the crew, each man receiving 1*l*. and $\frac{1}{100}$ part of the remainder; now if the number of men be added to the number of pounds each received, the square of the sum equals four times the value of the prize; required its value, the number of men, and the share of each?

Let x=*the number of men*, y=*the share of each, then will* xy=*the value of the prize, whence by the problem* $(\overline{x+y}|^2=)\ x^2+2xy+y^2=4xy$, *subtract* $4xy$ *from both sides, and* $x^2-2xy+y^2=0$, *the square root of this is* $x-y=0$, *whence* $x=y$, *consequently* $xy=x^2$=*the value of the prize; therefore by the problem* $1+\frac{x^2-1}{100}=x$ (*or* y)=*each man's share; whence* $\frac{x^2-1}{100}=x-1$; *divide this equation by* $x-1$ *and* $\frac{x+1}{100}=1$, *whence* $x=99$=*the number of men*, $y=(x=)$ 99 *pounds each, and* $xy=(99\times 99=)$ 9801*l*.=*the value of the prize*.

37. A learned society raised a fund for the purchase of a library; now if there had been 40 members more, each would have subscribed 3*l*. less than he did; but if there had been 50 less, each must have paid 6*l*. more: how many members were there, what did each contribute, and what sum did the library cost?

Let x=*the number of members*, y=*the sum each paid, then* xy=*the sum subscribed; whence* $\frac{xy}{x+40}$=*the sum each would have paid, had there been* 40 *more, and* $\frac{xy}{x-50}$=*the sum each must have paid, had there been* 50 *less; whence* $\frac{xy}{x+40}=y-3$, *and* $\frac{xy}{x-50}=y+6$, *by the problem*.

From the first of these $xy=xy+40y-3x-120$.

And from the second, $xy=xy-50y+6x-300$.

Whence $x=\frac{40y-120}{3}$ *and* $x=\frac{25y+150}{3}$, *therefore* $40y-$

$120=25y+150$, *or* $y=18l.$ $x=200$ *persons, and* $xy=3600l.=$ *the value of the library.*

38. A lady spent $\frac{4}{7}$ of her money at the linen draper's, $\frac{7}{9}$ of the remainder at the mercer's, $\frac{1}{2}$ of what she had left at the milliner's, paid 3 shillings for a hackney coach, and carried home $\frac{1}{30}$ of the sum she had at first; how much was it, what sum did she lay out at each place, and how much had she left?

Let $x=$ *her sum at first.*

Then $(\frac{4}{7}\times x=)\ \frac{4x}{7}=$ *sum paid the linen draper.*

And $(x-\frac{4x}{7}=)\ \frac{3x}{7}=$ *remainder.*

Then $(\frac{7}{9}\times\frac{3x}{7}=)\ \frac{x}{3}=$ *sum paid the mercer.*

And $(\frac{3x}{7}-\frac{x}{3}=)\ \frac{2x}{21}=$ *remainder.*

Then $(\frac{1}{2}\times\frac{2x}{21}=)\ \frac{x}{21}=$ *sum paid the milliner.*

Also $\frac{x}{30}=$ *sum carried home.*

Therefore $\frac{4x}{7}+\frac{x}{3}+\frac{x}{21}+\frac{x}{30}+3=x$, *by the problem.*

Whence $x-\frac{207x}{210}=3$, *and* $x=$ (210 *shillings*$=$) $10l.\ 10s.=$ *her sum at first. Whence also,*

$\frac{4x}{7}=(\frac{4\times 210}{7}=120$ *shill.*$=$) $6l.=$ *sum spent at the linen draper's.*

$\frac{x}{3}=(\frac{210}{3}=70$ *shill.*$=$)$3l.\ 10s.=$ *sum spent at the mercer's.*

$\frac{x}{21}=(\frac{210}{21}=)$ 10 *shillings*$=$ *sum spent at the milliner's.*

And $\frac{x}{30}=(\frac{210}{30}=)$ 7 *shillings*$=$ *sum carried home.*

39. A gambler lost at play $\frac{1}{4}$ of his cash, and then won 10 shillings; next he lost $\frac{1}{3}$ of what he then had, and won 10

shillings; lastly, he lost $\frac{1}{6}$ of what he had left, and went home with 45 shillings in his purse: what sum did he begin with?

Let x=*his number of shillings at first.*

Then $\frac{x}{4}$=*his first loss.*

And $(x-\frac{x}{4}=)\frac{3x}{4}$=*the remainder.*

Also $(\frac{3x}{4}+10=)\frac{3x+40}{4}$=*sum begun with the second time.*

Then $(\frac{1}{5}\times\frac{3x+40}{4}=)\frac{3x+40}{20}$=*his second loss.*

And $(\frac{3x+40}{4}-\frac{3x+40}{20}+10=)\frac{3x+90}{5}$=*the sum begun with the third time.*

Also $(\frac{1}{6}\times\frac{3x+90}{5}=)\frac{x+30}{10}$=*his last loss.*

Wherefore $(\frac{3x+90}{5}-\frac{x+30}{10}=)\frac{x+30}{2}=45$ *shillings by the problem; whence* $x=60$ *shillings=his sum at first.*

40. Out of a cask containing 81 gallons of wine, a quantity was drawn, and the cask filled up with water; the same quantity of the mixture was drawn off three several times after, and the cask filled up each time with water; after which it appeared, that the cask full of the mixture contained only 16 gallons of wine: how much wine was drawn off each time?

Let x=*the quantity of liquor drawn out each time.*

Then $81-x$=*the quantity left.*

But $(81 : 81-x :: x :)\ \frac{81x-x^2}{81}$ =*wine drawn the second time.*

And $81-x-\frac{81-x^2}{81}=)\ \frac{\overline{81-x}|^2}{81}$=*quantity left.*

Also $(81 : \frac{\overline{81-x}|^2}{81} :: x ;)\ \frac{\overline{81-x}|^2\times x}{\overline{81}|^2}$=*wine drawn the third time.*

And $(\frac{\overline{81-x}|^2}{81}-\frac{\overline{81-x}|^2}{\overline{81}|^2}=)\ \frac{\overline{81-x}|^3}{\overline{81}|^2}$=*quantity left.*

Then $(81 : \frac{\overline{81-x}|^3}{\overline{81}|^2} :: x :)\ \frac{\overline{81-x}|^3\times x}{\overline{81}|^3}$=*wine drawn the fourth time.*

And $\left(\frac{\overline{81-x}|^3}{\overline{81}|^2}-\frac{\overline{81-x}|^3\times x}{\overline{81}|^3}=\right)\frac{\overline{81-x}|^4}{\overline{81}|^3}=$ *quantity left.*

Therefore $\frac{\overline{81-x}|^4}{\overline{81}|^3}=(16=)\ \overline{2}|^4$ *by the problem.*

Whence $\overline{81-x}|^4=(\overline{2}|^4\times\overline{81}|^3=\overline{2}|^4\times 531441=)\ \overline{2}|^4\times\overline{27}|^4$.

Consequently $81-x=(2\times 27=)\ 54$, *and* $x=27$ *gallons.*

41. Borrowed of an usurer three sums of money: if the first be multiplied into the sum of the other two, the product will be 140,000 pounds; if the second be multiplied into the sum of the other two, the product will be 180,000*l.* and if the third be multiplied into the sum of the other two, the product will be 200,000*l.* what were the particular sums borrowed?

Let x, y, *and* z, *be the three sums respectively.*

$$\left.\begin{array}{l}\textit{Then } (x\times\overline{y+z}=)\ xy+xz=140000\\ (y\times\overline{x+z}=)\ xy+yz=180000\\ (z\times\overline{x+y}=)\ xz+yz=200000\end{array}\right\}\textit{ by the problem.}$$

The sum of these divided by 2, *gives* $xy+xz+yz=260000$. *Subtract each of the three former from this, and there arises,*

$$yz=120000,\ \textit{whence } y=\frac{120000}{z}.$$

$$xz=80000,\ \textit{whence } z=\frac{80000}{x}.$$

$$xy=60000,\ \textit{whence } y=\frac{60000}{x}.$$

But $(y=)\ \frac{120000}{z}=\frac{60000}{x}$, *consequently* $(120000x=60000z$, *or)* $x=\frac{1}{2}z$; *whence* $z=\left(\frac{80000}{x}=\right)\ \frac{80000}{\frac{1}{2}z}$, *therefore* $\frac{z^2}{2}=80000$, *or* $z^2=160000l.$ *whence* $z=400l.$ $x=(\frac{1}{2}z=)\ 200l.$ *and* $y=\left(\frac{60000}{x}=\right)300l.$

42. A merchant is indebted to A, B, C, and D, as follows: to A he owes half as much as to the other three, to B one third as much as to the other three, to C one fourth as much as to the other three, and to D 70 pounds less than to A; how much does he owe in the whole, and to each person?

Let the respective sums be x, y, z, *and* u; *then by the problem* $x=\frac{y+z+u}{2}$, $y=\frac{x+z+u}{3}$, $z=\frac{x+y+u}{4}$, *and* $u=x-70$; *whence*

$2x=y+z+u$, $3y=x+z+u$, $4z=x+y+u$, and $u=x-70$; add x to the first of these, y to the second, and z to the third, and we shall have $3x=(x+y+z+u=)\ 4y=5z$; whence $z=\frac{3x}{5}$, and $y=\frac{3x}{4}$; substituting these values respectively for z and y in the first equation, $2x=\frac{3x}{4}+\frac{3x}{5}+u$, whence $u=\frac{13x}{20}$; substitute this value for u in the fourth equation, and it becomes $\frac{13x}{20}=x-70$, whence $x=200l.$ $y=(\frac{3x}{4}=)\ 150l.$ $z=(\frac{3x}{5}=)\ 120l.$ and $u=(x-70=)\ 130l.$; also $(x+y+z+u=)\ 600l.=$the sum of the debts.

43. The difference of two numbers is 6, and their product 16; what are the numbers?

Let $x=$the greater, then $x-6=$the less, and $(\overline{x-6}\times x=)\ x^2-6x=16$ by the problem: complete the square, and $x^2-6x+9=(16+9=)\ 25$; whence by evolution $x-3=\pm5$, and $x=3\pm5=8$, or -2; also $x-6=(8-6$, or $-2-6=)\ 2$, or -8; wherefore the required numbers are 8 and 2.

Or thus,

Let $x=$the greater, then $\frac{16}{x}=$the less, therefore by the problem $x-\frac{16}{x}=6$; whence as before $x^2-6x=16$, and $x=8$, also $\frac{16}{x}=2$.

Or thus,

Let $x=$the greater, $y=$the less, then $x-y=6$, and $xy=16$. From eq. 1. $x=y+6$, substitute this value for x in eq. 2, and $(\overline{y+6}\times y=)\ y^2+6y=16$; whence by completing the square $y^2+6y+9=25$, and by evolution, &c. $y=2$, and $x=(y+6=)\ 8$, as before.

44. A pavement consists of 1000 equal square stones, there are 30 more in length than in breadth; how many are there in each?

Let $x=$the number in breadth, then $x+30=$the number in length, and by the problem, $(\overline{x+30}\times x=)\ x^2+30x=1000$: com-

plete the square, and $x^2+30x+\overline{15}|^2=(1000+\overline{15}|^2=)$ 1225; *by evolution* $x+15=35$, *whence* $x=20=$ *the number in breadth, and* $x+30=50=$ *the number in length.*

45. A rider on a journey received a sum of money, and afterwards as much more; on his return, having deducted 5*l.* for travelling expenses, he finds that the remainder is to the square of the sum first received as 3 to 64: what sum did he bring home?

Let $x=$ *the sum first received, then will* $2x-5=$ *the sum brought home; wherefore by the problem* $2x-5:x^2::3:64$, *whence* $(\overline{2x-5}\times64=x^2\times3$, *or*) $128x-320=3x^2$, *whence also* $x^2-\frac{128x}{3}=-\frac{320}{3}$; *complete the square, and* $x^2-\frac{128x}{3}+\frac{16384}{36}=(\frac{16384}{36}-\frac{320}{3}=)\frac{12544}{36}$; *and by evolution* $x-\frac{128}{6}=\pm\frac{112}{6}$, *whence* $x=(\frac{128\pm112}{6}=)\frac{240}{6}$ *or* $\frac{16}{6}=40l.$ *or* 2*l.* 13*s.* 4*d.* *whence also* $2x-5=(80-5$, *or* 5*l.* 6*s.* 8*d.* $-5=$) 75*l.* *or* 6*s.* 8*d.* *the former only answers the conditions of the problem.*

46. An army preparing a city to storm,
Is drawn up on the plain in rectangular form,
It contains sixteen thousand eight hundred brave men,
And the front exceeds triple the depth by thrice ten;
Say what number's in each, without blunder or fault?
And be quick—for the signal is made to assault.

Let $x=$ *the number of men in depth, then* $3x+30=$ *the number in front, and* $(x\times\overline{3x+30}=)$ $3x^2+30x=16800$ *by the problem, whence* $x^2+10x=5600$; *and completing the square,* $x^2+10x+25=5625$, *whence* $x=70=$ *the number in depth, and* $3x+30=240=$ *the number in front.*

47. Bought two pieces of mahogany for 20 shillings, one was two feet longer than the other, and each cost as many shillings per foot as it was feet in length; required the length of each?

Let $x=$ *the length of the less piece, then* $x+2=$ *the length of the greater; also* $(x\times x=)$ $x^2=$ *the value of the less, and* $(\overline{x+2}\times\overline{x+2}=)$ $x^2+4x+4=$ *the value of the greater, and the sum of these values* $=2x^2+4x+4=20$ *by the problem, whence* $x^2+2x=8$; *and completing the square* $x^2+2x+1=9$; *whence also* $x=2$ *feet* $=$ *the less piece, and* $x+2=4$ *feet* $=$ *the greater.*

48. A lady bought a basket of peaches for four shillings, but four of them proving bad, occasioned the rest to stand her in two-pence apiece more than they otherwise would have done; how many did she buy?

Let x *=the number, then* $x-4$ *=the number of good ones; also* 4*s*.=48 *pence=the sum they cost, therefore* $\frac{48}{x}$ *=the value of each had they all been good; and* $\frac{48}{x-4}$ *=the value of each of the rest,* 4 *being spoiled; whence by the problem* $\frac{48}{x-4}=\frac{48}{x}+2$; *this cleared of fractions, we have* $48x=48x-192+2x^2-8x$; *whence* $x^2-4x=96$, *and, completing the square, &c.* $x=12$ *=the number required.*

49. Bought eight cows, and sold them again for 56*l.* whereby I gained as much per cent. as they all cost; what sum did I give for each?

Let x *=the price of each, then* $8x$ *=what the* 8 *cost, and* $56-8x$ *=the gain on the whole; therefore by the problem* $8x : 56-8x :: 100 : (\frac{5600-800x}{8x}=)\frac{700-100x}{x}$ *=the gain per cent. whence by the problem* $8x=\frac{700-100x}{x}$, *or* $8x^2=700-100x$, *whence* $x^2+12.5x=87.5$; *and completing the square* $x^2+12.5x+39.0625=126.5625$, *whence* $x=5l.$ *=the sum each cost.*

50. Gave 20*l.* for a side board, and sold it again for a sum, which being multiplied into the gain, the product will be $\frac{3}{10}$ the cube of the said gain; what sum did it sell for?

Let x *=the gain, then* $20+x$ *=the sum it sold for; wherefore* $(\overline{20+x}\times x=)\ 20x+x^2=\frac{3x^3}{10}$ *by the problem, whence* $x^2-\frac{10}{3}x=\frac{200}{3}$; *this resolved, gives* $x=10l.$ *=the gain, and* $20+x=30l.$ *=the sum it sold for.*

51. Two gardeners carried between them 100 melons to market, and received equal sums; but A (who had the best) said to B, 'Had I carried as many as you, I should have received

$4l.\ 10s.$ for them;' says B, 'Had I carried no more than you, I should have taken only $2l.$ for mine:' how many had each, and what was the price?

Let $x = A$*'s number, then* $100-x = B$*'s, also* $4l.\ 10s. = 90$ *shill. and* $2l. = 40$ *shill. whence* $x : 100-x :: 40 : \frac{4000-40x}{x} =$ *the sum B received, and* $100-x : x :: 90 : \frac{90x}{100-x} =$ *the sum A received, whence* $\frac{4000-40x}{x} = \frac{90x}{100-x}$ *by the problem; this reduced, gives* $x = 40$ *melons* $= A$*'s number,* $100-x = 60$ *melons* $= B$*'s number; also* $(\frac{4000-40x}{x} \div 60 =)\ \frac{4000-40x}{60x} =)\ \frac{200-2x}{3x} = \frac{120}{120} = 1$ *shilling = the value of each of B's melons; and* $(\frac{90x}{100-x} \div 40 = \frac{90x}{4000-40x} =)\ \frac{9x}{400-4x} = \frac{360}{240} = \frac{3}{2} = 1$ *shill. 6 pence = the value of each of A's.*

52. Two partners gained $246l.$ A's money was 5 months in trade, and his share of the gain was $80l.$ less than his stock; B had put in $50l.$ less than A, but his money had been 7 months in trade; required the stock and gain of each?

Let $x = A$*'s stock, then* $x-50 = B$*'s stock; also* $x-80 = A$*'s gain, and* $(246-\overline{x-80} =)\ 326-x = B$*'s gain; now because A's stock × its time : A's gain :: B's stock × its time : B's gain, therefore* $5x : x-80 :: (\overline{x-50} \times 7 =)\ 7x-350 : 326-x$; *whence by multiplying extremes and means,* $1630x - 5x^2 = 7x^2 - 910x + 28000$, *or* $12x^2 - 2540x = -28000$, *or* $x^2 - \frac{635}{3}x = -\frac{7000}{3}$; *whence, by completing the square,* $x^2 - \frac{635}{3}x + \frac{100806.25}{9} = (\frac{100806.25}{9} - \frac{7000}{3} =)\ \frac{79806.25}{9}$; *whence* $x - \frac{317.5}{3} = (\pm\sqrt{\frac{79806.25}{9}} =)\ \pm\frac{282.5}{3}$, *and* $x = \frac{317.5 \pm 282.5}{3}$[f] $= 200l. = A$*'s stock,*

---

[f] It is evident that the affirmative value only must be used in this place, otherwise x would be $= 8\frac{1}{3}$; and B's stock, which $= x-50$, would be a negative quantity, which is absurd.

$x-50=150l.=B$'s stock, $x-80=120l.=A$'s gain, and $326-x=126l.=B$'s gain.

53. Of a company of boys playing at marbles, three won 50 between them, *B*'s winnings, if increased by the square root of *A*'s, would amount to 19, but if increased by the square root of *C*'s, the sum would be 21; how many did each win?

Let $x^2=A$'s winnings, $y^2=B$'s, and $z^2=C$'s; then by the problem $x^2+y^2+z^2=50$, $y^2+x=19$, and $y^2+z=21$; take the sum and difference of the second and third, and $2y^2+x+z=40$, and $z-x=2$ the square of the last is $z^2-2zx+x^2=4$; subtract this from the first, and $y^2+2zx=46$, this doubled, is $2y^2+4zx=92$: subtract the fourth from this, and $4zx-x-z=52$, but from the fifth $z=x+2$; this value substituted for z in the preceding equation, it becomes $(4x\times\overline{x+2}-x-x-2=)\ 4x^2+6x-2=52$, whence $x^2+\frac{3}{2}x=\frac{54}{4}$; this resolved, gives $x=3$, whence $x^2=9=A$'s winnings; also $z=(x+2=)\ 5$, whence $z^2=25=C$'s winnings; and $y^2=(19-x=)\ 16=B$'s winnings.

54. In a certain garden there are three square grass-plots, containing together 93 square yards, the difference of a side of the first and a side of the second is equal to the difference of a side of the second and a side of the third; moreover, if the side of the first be multiplied by 3, that of the second by 4, and that of the third by 5, the sum of the products will be 66; required the side of each?

Let $x-z=$ the side of the least, $x=$ the side of the second, then $x+z=$ the side of the greatest. The squares of these added together, give $3x^2+2z^2=93$; also $(\overline{x-z}\times3\times x+4+\overline{x+z}\times5=)\ 12x+2z=66$ by the problem, whence $z=33-6x$; this squared, is $z^2=1089-396x+36x^2$; this value substituted for z^2 in the equation $3x^2+2z^2=93$, gives $(3x^2+2178-792x+72x^2=)\ 75x^2-792x+2178=93$, whence $x^2-\frac{264}{25}x=-\frac{695}{25}$, whence $x=5$, $z=(33-6x=)\ 3$, wherefore $x-z=2$, and $x+z=8$.

55. A higgler bought two geese for 70 pence, and gave 8 pence more for one, than he did for the other; what did each cost him? *Ans. the first 3s. 3d. the other 2s. 7d.*

56. Two persons who are 60 miles apart, set out at the same time intending to meet: *A* travels 3 miles an hour, *B* 2; how

many miles and hours will each have travelled when they meet? *Ans.* 12 *hours. A travels* 36 *miles, B* 24 *miles.*

57. *A* is 5 years older than *B*, *B* is 4 years older than *C*, and the sum of their ages is 73 years; required the age of each. *Ans. A* 29 *years, B* 24, *C* 20.

58. In a christmas pudding, $\frac{1}{4}$ is flour, $\frac{1}{5}$ milk, $\frac{1}{6}$ eggs, $\frac{1}{3}$ suet and fruit, to which is added three quarters of a pound of spices and other ingredients; required the weight of the pudding? *Ans.* 15 *pounds.*

59. Three men bought a horse for 30*l.* *B* paid twice as much as *A*, and *C* paid as much as *A* and *B* together; what sum did each pay? *Ans. A paid* 5*l. B* 10*l. and C* 15*l.*

60. Some mice having made a lodgment in a Cheshire cheese, and nibbled away five shillings' worth, four-fifths of the remainder was sold for 2*l.* 16*s.* required the value of the whole cheese? *Ans.* 3*l.* 15*s.*

61. A courier was dispatched from Paris to Constantinople with orders to travel 20 miles a day; 6 days after, another was sent express after him, and travelled 35 miles a day; how many days must the latter travel to overtake the former? *Ans.* 8 *days.*

62. Two persons offered themselves as candidates for the office of schoolmaster in a certain village, where there were 329 voters, and the successful candidate gained his election by a majority of 53; how many voted for each? *Ans.* 191 *for the one, and* 138 *for the other.*

63. A vagrant, on being apprehended, gave the following account of himself, "I was apprenticed at 14, served 3 years, then ran away, have been $\frac{1}{12}$ of my life a sailor, $\frac{3}{8}$ a soldier, and have passed a year more than $\frac{1}{6}$ of my life subsisting mostly on charity;" how old was he? *Ans.* 48 *years.*

64. If from $\frac{1}{3}$ of my height in inches 12 be subtracted, $\frac{1}{5}$ of the remainder will be 2; what is my height? *Ans.* 5 *feet* 6 *inches.*

65. A footman, who was hired for 12*l.* a year and a livery, was turned away at the end of 7 months, and received 3*l.* 18*s.* 9*d.* besides his livery; what was its value? *Ans.* 7 *guineas.*

66. A bowl of punch was mixed as follows; $\frac{1}{6}$ was rum, $\frac{1}{4}$ brandy, $\frac{1}{12}$ acid and sugar, and 3 pints more than half of all these water; how much did the bowl contain? *Ans.* 6 *quarts.*

67. The paving of a square, at 2 shillings a yard, cost ten

times as much as the inclosing it at 5 shillings a yard; required the side of the square? *Ans.* 100 *yards.*

68. If *B* gives *A* 5 apples, *A* will have twice as many as *B*; but if *A* gives *B* 5, *B* will have thrice as many as *A*; how many apples has each? *Ans. A* 11, *B* 13.

69. Two gipsies bought a tin kettle, and as Sal paid ten pence more than Rachel, it was agreed that the latter should perform the duties of the kitchen five times to Sal's three; supposing this a just equivalent, what sum did the kettle cost? *Ans.* 3*s.* 4*d.*

70. Ten years ago, in manhood's early stage,
Two figures duly placed, expressed my age;
They'll do the same, when eight more years are past,
By placing that the first, which stood the last:
Time flies apace,—ye gay and thoughtless say,
Of mine, how many years have pass'd away? *Ans.* 34 *years.*

23
34
45
56
67
78
89

71. The weight of the head of Goliath's spear was less by one pound than one-eighth the weight of his coat of mail, and both together weighed 17 pounds less than 10 times the spear's head; required the weight of each? *Ans. the coat* 208 *pounds, the spear's head* 25 *pounds.*

72. The ages of a man and his wife are such, that his being multiplied into the square root of hers, the product is 180; but hers multiplied into the square root of his, gives 150 for the product; what are their ages? *Ans. the man's* 36 *years, his wife's* 25 *years.*

73. *A* shepherd with his flock was met by a company of soldiers, who plundered him of half his flock, and half a sheep over; a second, a third, and a fourth company met him, and plundered him successively of half what he then had, and half a sheep over, so that on arriving at home, he had but eight sheep left; how many had he at first? *Ans.* 143.

74. A trader maintained himself on 50*l.* a year, and cleared yearly one-third of his then remaining stock, by which means at the end of the third year his original stock was doubled; what sum did he begin with? *Ans.* 740*l.*

75. The sides of three cubical blocks of mahogany have equal differences, their sum is 15, and the solid content of the three together is 495 cubic inches; required the side of each? *Ans.* 3 *inches,* 5 *inches, and* 7 *inches.*

76. It is required to divide the number 27 into two such parts, that if the greater be divided by the less, and the less by the greater, the former quotient may be to the latter as 25 to 16. *Ans.* 15 *and* 12.

77. *A* is 4 years older than *B*, and the sum of the squares of their ages is 976; required their ages? *Ans. A's* 24 *years, B's* 20 *years.*

78. A nursery man planted 8400 trees at equal distances, in the form of a rectangle, having 50 trees more in front than in depth; required the particular number in each? *Ans.* 120 *in front,* 70 *in depth.*

79. A rectangular pavement consists of 533 equal square stones, and if the number in length be added to that in breadth, the sum will be 54; required the number in length and breadth? *Ans.* 41 *in length,* 13 *in breadth.*

80. What number is that which being tripled, and 15 added to the triple, the sum shall be to the square of the said number, as 6 to 5? *Ans.* 5.

81. A person bought mackerel for 30 pence, and receiving one more than the number originally bargained for, occasioned the whole to stand him in a penny a piece less than they otherwise would have done; what number did he buy? *Ans.* 6.

82. A lady bought lace for 4*l.* 4*s.* but on her arrival at home she found that 2 yards had been clandestinely cut off from the piece, whereby her lace cost a shilling a yard more than it ought to have done; how many yards were there? *Ans.* 14.

83. A gang of smugglers had their cargo seized, and were fined 1000l. on which two of the party absconded, leaving each of the rest to pay 25*l.* more than his proper share; how many were there in the gang? *Ans.* 10.

84. If my shillings be subtracted from 20, and if 6 be added to my shillings, the square of the remainder will be to the sum, as 8 to 11; how many have I? *Ans.* 16.

85. Two couriers set out at the same time from two cities 120 leagues apart, intending to meet; the first travels 5 leagues a day, the other 3 leagues a day less than the number of days they travelled; required the number? *Ans.* 10 *days*

86. The pile of a bridge is $\frac{1}{8}$ in the ground, 5 feet in the water, and the square of both together is to the part above water, as 27 to 5: required its length? *Ans.* 24 *feet.*

87. Bought 120 yards of ribband, and as many of binding,

and received one yard more of binding than of ribband for a shilling; now the ribband cost 6 shillings more than the binding; how many yards of each were bought for a shilling? *Ans. 4 yards of ribband, 5 yards of binding.*

88. A dealer bought a lot of pigs for 33*l.* 15*s.* and by selling them at 2*l.* 8*s.* each gained as much as a pig cost; what number did he buy? *Ans.* 15.

89. On the rumour of an approaching invasion, a nobleman, desirous of instructing his dependants in the manual exercise, endeavoured to form them into a square, but had 20 men too many; he then increased the side by one man, but wanted 21 to fill up the square; how many were there? *Ans.* 420.

90. A rectangular garden contains 1200 square yards, and the length is to the breadth as 4 to 3; what will the fencing cost at 3*s.* 6*d.* per yard? *Ans.* 24*l.* 10*s.*

91. A farmer bought a number of calves for 40*l.* and after reserving 5, sold the rest for 36*l.* whereby he gained 8 shillings a head; how many did he buy? *Ans.* 20.

92. The maypole of a country village being blown down, was broken into three pieces by the fall; the common difference of their lengths was 1 foot, and the cube of the greater piece exceeded the product of all three by 560; required the length of the pole? *Ans.* 39 *feet.*

93. A joiner has a mahogany plank 10 feet long, and is desirous of cutting it into two unequal segments, each of which is to constitute the side of a square table, the less table to be $\frac{2}{3}$ of the greater; whereabouts must he cut the plank? *Ans. at* 4.494897 *feet from the end.*

94. *A*, *B*, and *C* freight a ship, towards which *B* pays 100*l.* more than *C*, and *A* 200*l.* more than *B*, moreover *A*'s payment is equal to thirty times the square root of the sum of *B*'s and *C*'s; what sum does each pay? *Ans. A* 1500*l.* *B* 1300*l.* *C* 1200*l.*

95. If the number of people in a certain parish be quadrupled, and 13000 added to the quadruple, the sum will be to the square of the number as 3 to 50; how many does that parish contain? *Ans.* 500.

96. Sold a quantity of almonds for 6*l.* and gained four times as much per cent as the almonds cost; how much was that? *Ans.* 5*l.*

97. Two hawkers, *A* and *B*, travel together, *A* has 40 ells of cloth, *B* 90; *A* sells $\frac{1}{3}$ of an ell more for a crown than *B*; hav-

ing sold the whole, they find that they have taken 42 crowns between them; how many ells did each sell for a crown? *Ans. A sold* $3\frac{1}{2}$ *ells, B* 3 *ells.*

98. Two merchants dissolving partnership, shared 70000*l.* between them; *A*'s share of the original stock was 12000*l.* and *B*'s share of the gain 24000*l.* what was *B*'s share of the stock, and *A*'s gain? *Ans. B's share* 16000*l. A's gain* 18000*l.*

99. What number is that which being divided by the product of its two digits, the quotient is 2, and if 27 be added to the number, the digits will be inverted? *Ans.* 36.

100. A nobleman hired a painter for 42 days, and agreed to give him as many shillings for his trouble, as would be equal to half the square of the number of days he worked; and to deduct as many as would equal the square of one-third the number of days he was idle; the time being expired, the painter received at the rate of six shillings a day for the whole time; how many days did he work, and how many was he idle? *Ans.* 24 *days worked, and* 18 *days idle.*

101. The joint stock of two partners was 3000*l.* *A*'s money was 12 months in trade, and *B*'s 9 months; when they shared stock and gain, *A* received 1984*l.* and *B* 3476*l.* what was each man's stock and gain?

$$\textit{Ans. A's} \begin{cases} \textit{stock } 1000l. \\ \textit{gain } \;\; 984l. \end{cases} \quad \textit{B's} \begin{cases} \textit{stock } 2000l. \\ \textit{gain } \; 1476l. \end{cases}$$

102. The ages of two persons are such, that the age of the first is to that of the second, as that of the second is to 50, and the sum of the squares of their ages is 2624; required the age of each? *Ans. the first* 32, *the other* 40.

103. A trader at the end of the first year had doubled his original stock, the second year he gained 80*l.* more than the square root of his said increased stock, the third year being very successful, he cleared half the square of all he had at the end of the second, and leaving off business, he finds himself worth 18240*l.* what sum did he begin with? *Ans.* 50*l.*

END OF VOL. I.

Printed by Bartlett and Newman, Oxford.